

Activities to Make Learning Functions More Comprehensible

THESIS

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ABSTRACT

Most of today's high school students struggle in understanding one or more of the essential and standard elements taught in high school mathematics classes. Regardless of the cause, this lack of understanding translates to a poor mathematical foundation and thus a lack of understanding in other high school mathematics as well as in future careers. One of the primary areas in high school mathematics with which many students struggle and which many commonly misinterpret is the concept of functions and the accompanying ability to manipulate and utilize them.

To eliminate some of the frustration for the high school students, as well as for the teachers, I have developed supplementary materials and outlined activities that clearly explain and analyze the concept of functions partitioned in its component parts. Some of the education and human development philosophers whose ideas have influenced the activities I have compiled include Bloom, Glasser, Albert, Gardner, Wolfe, Cohen, and Dunn. The work of these theorists and current research on the subject of how students learn best have provided me with ideas and methods to follow in creating the supplementary materials.

This thesis strives to make functions more comprehensible by thoroughly covering all that the high school student would be expected to learn about functions in an understandable way.

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I. INTRODUCTION

Many of today's high school students struggle in understanding one or more of the essential and standard elements taught in high school mathematics classes. Poor understanding of preliminary mathematical concepts can inhibit learning new mathematical content. Various factors, including social and economic circumstances, affect student learning and retention. Regardless of the cause, this lack of understanding of preliminary mathematical concepts translates to a poor mathematical foundation, which hinders more advanced conceptual understanding in middle school and high school mathematics, as well as in future careers (National Council of Teachers of Mathematics, 2000). Frustration caused by lack of understanding, causes many students to strongly dislike math by the time they get to algebra. Sadly, the added frustration in algebra leads many students to dislike math to an even greater degree, creating another barrier between themselves and comprehension (National Council of Teachers of Mathematics, 2000). The algebra teacher has no control over the past, but success in this subject is still possible (Glasser, 1969).

According to many high school mathematics teachers, one of the primary areas in high school mathematics with which many students struggle and which many commonly misinterpret is the concept of functions and the accompanying ability to manipulate and utilize them (J. Garred, personal communication, September, 2004; J. Handsbury, personal communication, April, 2005; C. Horn, personal communication, April, 2005; M. McAllister, personal communication, April, 2005; J. Plewa, personal communication, April, 2005; S. Shockey, personal communication, April, 2005; C. Vaughn, personal communication, September, 2004). Furthermore, functions are the foundation of not only algebra but also all future mathematics (National Council of Teachers of Mathematics, 2000). Without this foundation firmly in place, it is no wonder students are struggling in higher levels.

This thesis was constructed in hopes of eliminating some of the frustration for the high school students as well as for the teachers. This project presents supplementary materials and outlined activities that clearly explain and analyze the concept of functions partitioned in its component parts. It covers all that the high school algebra student would be expected to learn about functions in an understandable and interactive way.

The topic of how students learn and techniques to help students retain information better has long been the subject of study of educational psychologist and theorists. One such theory is constructivism, which maintains that, in order to learn well and retain their learning, students must construct their own understanding from experiences (Brooks & Brooks, 1993). Another theory of learning maintains that students must be taught material that is in their *zone of proximal development*, and the teacher must use careful scaffolding to increase the students' schema to encompass the new concepts presented (Bruer, 1999, Vygotsky, 1997, Wolfe, 2001). The activities in this thesis are designed to utilize the constructivist approach through a carefully sequenced set of activities that allow students to build understanding of functions from simple concepts (Brooks & Brooks, 1993); the activities, furthermore, are designed to help students master these function concepts by teaching them concepts that are just slightly beyond their current learning level, thus offering scaffolding and working with their zones of proximal development (Bruer, 1999, Vygotsky, 1997, Wolfe, 2001).

The activities are also designed to appeal to students who learn best through auditory, visual, kinesthetic, or tactile learning modalities (Guild & Garger, 1998), and to students who may manifest one or more of Howard Gardner's multiple intelligences, specifically, musical, linguistic, spatial, bodily-kinesthetic, naturalist, intrapersonal, interpersonal, and/or logical-mathematical intelligences (Gardner, 1983). The varied activities utilize these differences in an attempt to reach different students, prevent boredom, and attend to the environment in that it can have such an important effect on the way students learn (Dunn & Dunn, 1975).

In an attempt to make the activities in this thesis useful, certain other qualifications were followed. One of the qualifications of the activities is that they not only aid comprehension but also allow the normal pace of the class to continue. The activities do not take a lot of time to explain or do; thus a teacher may fit them into a forty-five minute period or an hour and a half period. Algebra teachers are required to cover a great number of concepts and processes, and seldom are able to cover all the elements they are expected to cover (J. Garred, personal communication, Spring, 2005; J. Handsbury, personal communication, April, 2005; C. Horn, personal communication, April, 2005; M. McAllister, personal communication, April, 2005; J. Plewa, personal communication, April, 2005; S. Shockey, personal communication, April, 2005; C. Vaughn, personal communication, September, 2004); these activities are designed to lessen the time required for comprehension of functions and help the overall curriculum flow better.

These activities are directed toward the whole class; however, to help the students who are really struggling, the teacher will of course need to spend individual time and give extra care. These activities are not for the struggling student; they are designed to prevent so many students from struggling with the concept of functions by breaking functions into components, broadening the picture, relating functions to real life, or something students find relevant, and making learning more interactive.

Further qualifications considered to make the content of this thesis more practical are activities with relatively inexpensive or readily available materials, experiences geared toward the entire classroom group, and content consistent with the National Council of Teachers of Mathematics (NCTM) Standards (National Council of Teachers of Mathematics, 2000). The NCTM standards covered are specified at the beginning of each set of activities. These qualifications will help ensure this thesis' effectiveness in the classroom setting.

This project was inspired by, and is the result of, many conversations with high school mathematics teachers, college professors, scientists, and students regarding the typical struggles with mastering of the function concept. Additional ideas were spawned from articles written in

educational journals, current algebra and educational textbooks, media sources, the Internet, and elsewhere. The activities were designed to follow the recommendations of well-respected learning theories and theorist.

The order of the activities is very specific in that the ideas learned in earlier activities are necessary to complete the later. Therefore, first are the activities relating to concept of domain and range. These can be presented understandably regardless of previous mathematical foundation. Before any understanding of functions can take place the students must have some concept of domain and range. Students must know not only what domain and range are, but also how they apply to functions since domain and range play such a critical part not only in the definition of functions but also in any operations done on or with the function.

Next the students must understand the basic definition of a function and its applicability so that they feel a desire to learn the concept (Glasser, 1988), and understand what exactly they are being asked to think about. Hence, basic definition of functions and applicability thereof is the focus of second set of activities.

The next set of activities regards the different ways that functions can be manipulated and why this is significant and a desirable thing to do. This includes finding results of different values, the basic operations that can be done with functions, and the different areas where function manipulation is used.

Composition of functions is so significant a concept and so essential for understanding higher levels of math that it needs a set of activities all of its own to clarify not only the process, but also its implications. The fourth set of function activities also focuses on the concept of finding and checking the inverse of functions, which cannot be understood until composition of functions is clear.

The final set of activities considers the different families of functions and how they relate to each other. This, like the other sets of activities, ties into the sets of activities presented previously. Families of functions ties into domain and range, the definition of function, function

manipulation, and function composition. The different families of functions include all the polynomial families, the absolute value family, the exponential families and logarithmic families.

There is much more about functions not covered in this thesis due to the focus being high school algebra students. These five components of functions translated to five sets of activities and the five major sections of this work. Each activity is either an attempt to simplify the concept, to make it more interesting or interactive to reach different learning styles, or to explain it in a new more applicable way. Most activities are focused not on rote memorization, repetition, and calculation but on understanding and explanation. Many of the activities require additional handouts or materials; when possible these are included in the appendix.

II. DOMAIN AND RANGE

Goal: To help students understand the concepts of domain and range as well as find the domain and range in varied situations.

A. Standards:

NCTM Algebra Standards for grades 9-12

Standard 1: Understand Patterns, relations, and functions.

b) Understand relations and functions and select, convert flexibly among, and use various representations for them. (National Council of Teachers of Mathematics, 2000, p 395)

B. Background:

To understand functions, students must have a very clear understanding of domain and range. This can be presented in many different ways and it is important that the student understand the broadness of this concept. This set of activities only focuses on domain and range and the different notations for writing domain and range because it is from this understanding that the definition of functions is built (National Council of Teachers of Mathematics, 2000).

One of the first places to discuss domain and range is in a relation. In a relation, the second element is not always dependent on the first element in the function sense; it only means that you look at the first element first and draw the conclusion from it, or even that you just look at the first element before the other. Selecting students and forming a set of ordered pairs by writing their eye color before their hair color forms a relation. The eye color then is the domain and hair colors the range even though the eye color does not determine the hair color; it is the student that determines both. A relation is a function when where you start from can only end in one place, for example, if everyone with brown eyes only had brown hair, and blue-eyed people only had blond hair. Because this is not true, eye color vs. hair color is not a function, but it is still a relation.

The next place to discuss domain and range is graphically. Graphs very often represent real world situations; and understanding what is practical for input and output values in terms of graphs is important. For example, often area is presented in the form of quadratic graphs with

the single dimension on the x axis and the area on the y axis. Finding the vertex of a parabola can be used to find the maximum area, but only values in the first quadrant are valid because neither length nor area can be negative, therefore area is represented by a bounded graph. By considering the concepts of domain and range both algebraically and graphically a full understanding can be reached.

Activity 1: Domain? Range? (Initial definition of domain and range especially in regard to relations)

By using ideas that the students come up with, this activity focuses on an understanding of the definitions of *domain*, *range*, *relation*, *independent*, and *dependent*, and teaches the set form of writing the domain and range of a relation.

Materials: Overhead and student paper and pencil for notes.

Step 1: Vocabulary lesson.

A *relation* can be any set of pairs of objects that are “related” in anyway. More formally, a *relation* is a set of ordered pairs. To teach the vocabulary and prepare for the next part, have the students generate a list of things that are related in some way including some cause and effect relationships. Here are some examples to help get the students started:

- Hair length and time it takes to shower
- Parents and children
- Time of year and temperature
- Time studying and test grade
- Age and favorite music
- School and mascot
- Picture books and novels

- School and age
- Time practicing and skill
- Extra curricular activities and free time
- Eye color and hair color
- Name and number of siblings
- Pets and pet names
- Weight and eating/ exercise habits
- Percent grade and letter grade
- Age and height
- Salary and occupation
- Total pay and hours worked

Step 2: More vocabulary and a little review

Remind the students which element in each ordered pair is the x-coordinate, and which is the y-coordinate (alphabetical order). Tell them that there are several names for each coordinate. X-coordinate = first coordinate = domain element = independent variable = input. Y-coordinate = second coordinate = range element = dependent variable = output. More formally, the *domain*, D , is all possible values of x (or input values) and the *range*, R , is all possible values of y (or output values).

Some additional suggestions to help students remember:

- Compare order in the alphabet. Domain and range are also alphabetical like the x and y and the words first and second.
- Make words out of the three names, “in-domain- x ” and “de-range- y ”

It is always useful to help students understand what the new math vocabulary words mean in plain English before they are applied to math terms. By high school the students should be familiar with the words and if these words are not part of their English vocabulary they will at least have heard the words a sufficient amount of times to be able to build on their prior

knowledge. This will help the students understand why the vocabulary was chosen the way it was.

For example with domain and range, *domain is where one comes from; range is where one goes*. Domain is where an organism lives, or where it originates. A person's domain could be their house, or a student's domain could be their own desk in your classroom. Range means where the organism travels. For example, students go to school, the mall, church, or to the grocery store. In other words, the domain, x , is where you start from, the beginning. The range, y , is where you go, the end or the finish line. Sometimes domain and range can be the same or someplace or something switches from a domain to a range. For example you can start at home (domain) go to church (range) then from church (new domain) to the grocery store (new range). Once the students understand the words the switch from places to objects and even to numbers is easy.

Students also need to understand the words independent and dependent in English before they are applied to math terms. Independent means self-sufficient, self-supporting, or self-reliant. Dependent means reliant on someone or something else, not self-supporting. In terms of people one has to be independent (live on their own and support themselves) before they should have dependents (children). Most students will have heard of parasites. They are completely dependent on the creature they feed live and feed on. Applied to math, and domain and range, you cannot have y without x first, or x creates/ makes/ changes/ determines y , or y is reliant/ contingent on x .

Step 3: Go back to the list once more.

Revisit the list that the students have helped to create. Decide whether each relation is either an independent/ dependent relationship or not. If it is an independent/ dependent relationship decide which variable is x (independent) and which is y (dependent). If the relation is not an independent/ dependent relation, still assign variables but it will not matter how. Here are some examples explained in detail:

- Letter grade is dependent on the number/ percent grade. This means the letter grade being dependent is y , and the number grade being independent is x .
- Grades on a test are dependent (y) on the amount of time studying for a test (x).
- A person's weight is dependent (y) on how well they take care of themselves and their genes (x).
- The temperature outside (y) is dependent on the time of year, the wind patterns, and/or other factors (x).
- How you smell is dependent (y) on when you shower, what you wear, and/or what you do(x).
- The picture book is not dependent on the novel nor is the novel dependent on the picture book so picture book could be either x or y and novel the other variable.

Step 4: Cover the more formal definition of relation (a set of ordered pairs).

After they have come up with a list of things that are a relation, and determined independent/ dependent relationships, have them pick a few examples off the list and as a group turn them into a relation in the form of a set of ordered pairs. Here are two examples:

Time studying in hours (x -coordinate) and test grade in percent (y -coordinate) a possible relation could be: $\{(0, 23), (1, 72), (1, 82), (1.5, 74), (3, 94)\}$

Age in years (x -coordinate) and favorite music type (y -coordinate) a possible relation could be: $\{(1, \text{mom singing}), (2, \text{children songs}), (3, \text{children songs}), (5, \text{oldies}), (9, \text{children's pop}), (12, \text{country}), (15, \text{rock})\}$

Step 5: Write the domain and the range separately using the set notation.

By using examples that the students come up with and speaking of domain as the starting point, the independent, or the cause, and the range and the ending point, the dependent, or the effect, the students should come to a quick understanding of domain and range and be able to easily list the separate elements in the set notation.

As a class, divide the set of ordered pairs in to two sets, one for the domain and one for the range. In the studying example, the domain of the relation is the number of hours studied for the test, and the range is the test grade. Let them know that each element only needs to be written once, so even though there were two 1's, "1" only needs to be written once in a set.

$$D: \{ 0, 1, 1.5, 3 \}$$

$$R: \{ 23, 72, 82, 74, 94 \}$$

In the music example, domain is the age and range is the favorite type of music.

$$D: \{ 1, 2, 3, 5, 9, 12, 15 \}$$

$$R: \{ \text{mom singing, children songs, oldies, children's pop, country, rock} \}$$

Activity 2: Find it! (Finding domain and range with graphs and using different notations)

This activity uses a set of different graphs including functions, continuous and non-continuous, and graphs that are not functions to focus on a graphical representation and the varied notations used to write of domain and range. The variety of different types of graphs as well as the amount of graphs used is important to insure enough practice. Also, this is meant as a small group activity with a lot of supervision because of the difficulty in finding domain and range accurately. Without supervision there is a chance that the students will misunderstand and reinforce the idea in their mind incorrectly.

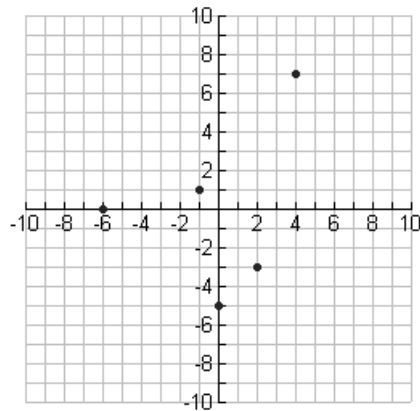
Materials: A set of overhead transparencies with different graphs on them. Included in the appendix is a possible set titled "Find It". Students should have writing implements and paper.

Step 1: Demonstrate and teach the different notations in which domain and range can be written.

Start with four simple examples: a graph of relation, a graph of bounded line, graph of continuous infinite function (line or otherwise), and a graph with a discontinuity (either an

asymptote or a hole). In each example write the same domain and range in all of the notations in which it can be written. For the relation, only the set notation is applicable. For the bounded line and continuous functions the domain and range can be written in the interval notation and the inequality notation. For the graph with a discontinuity the domain and range can be written in the interval notation, the exclusion notation, or in the inequality notation.

An example of a relation:



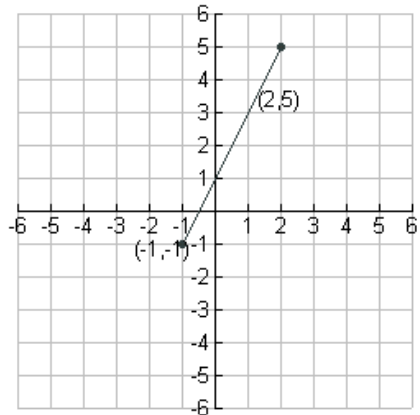
Relation: $\{(-6,0), (-1, 1), (0, -5), (2, -3), (4,7)\}$

D: $\{-6, -1, 0, 2, 4\}$

R: $\{0, 1, -5, -3, 7\}$

} Set Notation

An example of a bounded line:

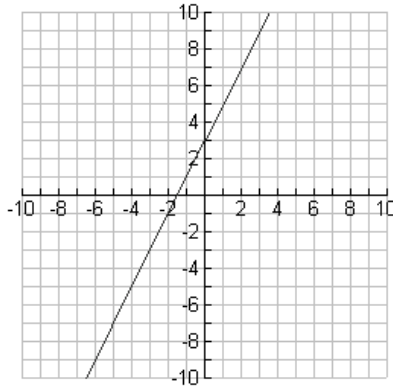


$y=2x + 1$ bounded by the points $(-1,-1)$ and $(2, 5)$.

$$\left. \begin{array}{l} D: -1 \leq x \leq 2 \\ R: -1 \leq y \leq 5 \end{array} \right\} \text{Inequality Notation}$$

$$\left. \begin{array}{l} D: [-1, 2] \\ R: [-1, 5] \end{array} \right\} \text{Interval Notation}$$

An example of a continuous function:

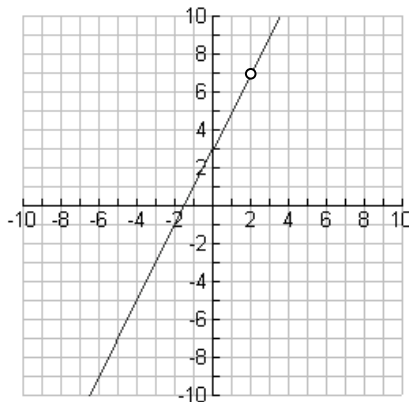


$$y = 2x + 3$$

$$\left. \begin{array}{l} D: -\infty < x < \infty \\ R: -\infty < y < \infty \end{array} \right\} \text{Inequality Notation}$$

$$\left. \begin{array}{l} D: (-\infty, \infty) \\ R: (-\infty, \infty) \end{array} \right\} \text{Interval Notation}$$

An example of a graph with a hole:



$$y = 2x + 3 \text{ with a hole at } (2, 7)$$

D: $x < 2$ or $x > 2$, or D: $-\infty < x < 2$ or $2 < x < \infty$ }
 R: $y < 7$ or $y > 7$, or R: $-\infty < y < 7$ or $7 < y < \infty$ } Inequality Notation

D: $x \neq 2$ }
 R: $y \neq 7$ } Exclusion Notation

D: $(-\infty, 2) \cup (2, \infty)$ }
 R: $(-\infty, 7) \cup (7, \infty)$, } Interval Notation

Issues to bear in mind:

- It is important to say what every symbol means as it is being written. Verbal descriptions will help as well as the symbolic representation, also verbal repetition will enforce the vocabulary.
- For the interval method, intervals look a lot like ordered pairs, but they are different; because of this it is important that the students write the “D” and the “R” in front of the interval to define it, or actually write “defined on the interval...”
- Clarify from the beginning the relationship between the brackets “[” or “]” meaning “including”, or “equal to” and the symbol \leq or \geq , and the relationship between the parentheses “(” or “)” meaning “not included” or “not equal to” and the symbols $<$ and $>$.

Step 2: Divide the class into small groups

The task of each group is to correctly find the domain and range of the graph, write it in one of the proper notations, and make sure that all in the group understand.

Step 3: Put the graphs on the overhead one at a time.

Instruct the groups to find the domain and range of the graph. After a few minutes have each group write their answer on the board one at a time. If their answer is already up there, have them put a check by it. Monitor the groups to make sure they are attempting every problem and not just waiting for an answer to be put on the board.

Step 4: Verify understanding, congratulate effort, and reinforce the different notations, then repeat.

After all the options are on the board go through them as a class. Answer any questions.

Continue the process for all the graphs.

Activity 3: Restricted domain? (Discovery activity: solving for domain and range algebraically)

Using multiple examples students will, in the format of a discovery activity, explore equations with limited domains and attempt to determine algebraic rules for finding the domain. Because in working with functions the domain determines the range, and because most functions the students will be working with at this stage (algebra 1) have an infinite range, this activity focus on finding only the domain algebraically. The two things in this level that limit the domain of an equations, specifically functions, are dividing by zero because this is undefined, and having a negative underneath an even root because that produces an imaginary number. Though in other equations, such as circles and hyperbolas, there are other reasons for the domain to be limited, mathematically they follow the same rules.

Materials: Graphing calculator for every student. Detailed instructions on calculator process with the TI 83 are in the Appendix titled “Restricted Domain?”.

Step 1: Make a long list of functions with limited domains.

Some examples of limited domain functions:

$$y = \sqrt{x}$$

$$y = \sqrt{x+3}$$

$$y = 2\sqrt{3x-1}$$

$$y = x\sqrt{x-2}$$

$$Y = \frac{x-1}{2x+1}$$

$$y = \frac{3}{5-x}$$

$$y = \frac{3x}{x+2}$$

$$y = \frac{-3}{-x+2}$$

$$y = \frac{6}{x(4+x)}$$

$$y = \frac{x^2+2x}{-x-3}$$

$$y = \frac{3x}{x+2}$$

$$y = \frac{-8}{x(x+5)(x-1)}$$

Step 2: Class meditation on limited domains.

If the students have already done the first two activities, or something similar, they know about limited domains. Put a limited domain function on the overhead for the students to view while you are discussing limited domains. Tell the students something to the effect of “there are some numbers that cannot be plugged in for x or numbers that x can never be. This means that those numbers are not part of the domain of the equation.” Tell the students to think about it, but not say anything yet.

Step 3: Divide the students into small groups and teach the process with the equation on the overhead.

Each group is to enter the equation into the graphing calculator and view the graph to predict what values will not work or what values x cannot be. They are to write down a few ideas in the first column of a table. Then use the “function ability” of the calculator to try different values for x . (Using this ability will help prepare the students for the activities in the function manipulation set because it uses correct notation and introduces them to the word *function* as plugging in values.) This is a way to try different values in the functions quickly. Have the students make a chart of the values that work, and those that do not (result in an *error* on the calculator) by keeping track with a table. For example for the function $y = \sqrt{x}$

x	$f(x) \rightarrow y/ n$
3	yes
0	yes
-1	no
-4	no
-6	no

So $x \geq 0$

Monitor the groups carefully and give hints of possible numbers that will not work when necessary to reduce frustration. When they think they see a pattern have them write the pattern as a statement of the domain of the function in one of the notations they have learned. Have them check the domain with you.

Step 4: When they have solved it correctly give them another limited domain equation and have them follow the same process.

Each group will be able to work at their own pace. Do not give the group another equation until they have figured out the one they already have. Give the groups several of each type of limited domain problem. Ask if they can discover any similarities in the equations and the corresponding domain. Have the students look for patterns, and see if they can explain the patterns.

Step 5: As a wrap up, go through the equations you have given the students.

With the examples the students have already worked with show them how to solve for the domain algebraically.

For example, with the even root equations set the value under the root to greater than or equal to zero. For the equation $y = \sqrt{x+3}$; $x+3 \geq 0$, so $x \geq -3$. So D: $x \geq -3$ or D: $[-3, \infty)$.

Once again remember there are many correct way to write the domain. Let the students use the method with which they are most comfortable. For the equations with a denominator, set the denominator not equal to zero. $Y = 3/(x-2)$; $x-2 \neq 0$; $x \neq 2$. So D: $x \neq 2$, $x > 2$ or $x < 2$, or D: $(-\infty, 2) \cup (2, \infty)$.

III. FUNCTION DEFINITION

Goal: To help the student understand the definition and purpose of a function.

A. Standards

NCTM Algebra Standards for grades 9-12

Standard 1: Understand Patterns, relations, and functions.

- b) Understand relations and functions and select, convert flexibly among, and use various representations for them.

Standard 2: Represent and analyze mathematical situations and structures using algebraic symbols.

- a) Understand the meaning of equivalent forms of expressions, equations, inequalities, and relations
- b) Write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency- mentally or with paper and pencil in simple cases and using technology in all cases.
- c) Use symbolic algebra to represent and explain mathematical relationships
- d) Use a variety of symbolic representations, including recursive and parametric equations, for functions and relations
- e) Judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.

Standard 3: Use mathematical models to represent and understand quantitative relationships

- a) Identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships
- b) Use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts.
- c) Draw reasonable conclusions about a situation being modeled. (National Council of Teachers of Mathematics, 2000, p 395).

B. Background

In order to make the lesson of functions applicable to the students, it is important that they understand how the concept of a function exists outside of the mathematics classroom.

Some real world examples of functions and some examples with numbers using those functions could help.

How do you define “real life”? Does it have to be part of a kid's everyday experience outside of school, as opposed to something engineers do every day to design the devices kids use every day? I think “real life” is a lot bigger than most students realize! And most interesting applications of math occur in the context of other math, science, or engineering, rather than in the ordinary activities of life, so it's not realistic to pretend everyone will be using advanced math whenever they go shopping or something, or that anything outside their experience is unimportant (Ask Dr. Math, 2005).

But it is important to make the examples as applicable and understandable as possible.

Here are some examples just to help students see how applicable and important functions are.

One function students are very familiar with is grades. The number grade that they get is functionally related to the letter grade they get. It can be written as a piecewise function or it could be partially written as a relation, example: $\{(54, F), (62, D), (69, D), (93, A), (76, C), (88, B)\}$.

A simpler example of a one variable function is a dimmer switch or the volume on a stereo. The volume or the amount of light showing is dependent on the number that the dial is sitting on and the amount will always be the same as the corresponding number. $f(\text{dial position}) = \text{amount of light}$.

To make sure the students understand the concept of negative numbers and how they relate to functions and relations, you could suggest a function involving spending. For example if they start off with \$2 and want to keep buying cokes for \$1 each; the function could look like this $M = -1C + 2$ producing a relation like $\{(0, 2), (1, 1), (3, -1), (5, -2)\}$. They will have to keep borrowing money to support their caffeine addiction. Another familiar function is that the amount of hours someone works is functionally related to the amount they get paid. $P = \$5.50h$. An example of a constant function could be the amount that people pay to get into the movies versus their age, $P = \$7.50$.

Functions more frequently exist in more complicated situations and usually involve more than one variable. It may be useful to present this concept to help students appreciate the beauty of the concept and the simplicity of what you are asking them to do. A good example of a real world function is a vending machine. $\text{Candy}, f(\text{money}, \text{letter}, \text{number}) = \text{candy bar, or coke}$, $f(\text{money}, \text{button}) = \text{soda}$. The inputs have to be given in the right order, as well as the right amount, otherwise a solution of nothing will appear instead of a specific candy or soda. These give the students the big idea of functions. An explanation to follow could include that for

current purposes only single variable functions will be used, and since most real world functions are multivariable that many future examples would not be “real world”.

More functions in real life include video games, barcodes, microwaves, sewing machines, computers, and washing machines. In the microwave you put in food and a number and the output is that food cooked a certain amount. In the washing machines the input is the dirty clothes, the output is the same clothes clean.

For contrast some examples of things that are not functions may be helpful. For example slot machines are not functions, the same amount of money and pulling the handle will not always produce the same sequence of three symbols. Wind chimes are not a function; they make a different song every time the wind blows, neither is firing a gun in real life as opposed to video games a function. The point is that there are too many inputs that cannot be controlled for these things to act as practical functions.

Activity 1: Ball Game (Given a relation determine if the relation is a function non-graphically – exploring the definition of functions)

The purpose of this activity is to teach students the definition of a function by having them act it out. It is one thing to hear the definition and copy it down, it is another thing entirely to be part of a function and see why they are important and how they work.

Materials: Balls with numbers on them for the students to throw, boxes with numbers on them for the balls to be thrown in. This same activity could be done by just numbering your students and putting circles on the floor or blocking off different sections of the room for the students to move to.

Step 1: Write a relation that is a function on the overhead and define x and y .

The students should already be familiar with domain and range and with relations. This relation should not only be a function but also have every ball go into a different box. Define the x-coordinates as the numbered balls or the students and the y-coordinates as the numbered boxes or the circles on the ground or places in the room.

Ex. $\{(2,1), (3,4), (5,6), (9,3), (10, 5), (13, 2)\}$ (for 15 balls and 6 boxes)

Step 2: Perform the function.

Let the students perform the function by throwing their ball in the right box for example: $\{(2,1), (3,4), (5,6), (9,3), (10, 5), (13, 2)\}$ ball 2 is thrown in box 1, ball 3 is thrown in box 4, ball 5 is thrown in box 6, etc. Tell the students this relation is a function, ask them to guess why- answer: because it was possible to throw each ball in the specific box.

Step 3: Write another relation that is a function.

This time write the relation so that more than one ball goes into the same box. Example: $\{(1,2), (4,1), (6,3), (7, 5), (8, 4), (9, 6), (11,3), (12, 2), (14, 3), (15, 1)\}$, ball 1 goes into box 2 and also ball 12 goes into box 2. Tell the students this is a function as well; see if any of their guesses change.

Step 4: Write a relation that is not a function.

Example: $\{(2, 4), (6, 1), (3, 1), (8, 4), (9, 2), (10, 3), (11, 5), (1, 3), (4, 3), (8, 3)\}$ – this is not a function because person number 8 will not be able to throw their ball into both box 4 and box 3.

Example: $\{(7, 6), (5, 5), (9, 6), (12, 3), (14, 2), (13, 2), (15, 1), (5, 6)\}$ - this is not a function because person number 5 will not be able to throw their ball in both box 5 and in box 6.

Step 5: More examples.

Use several more examples of both relations that are functions and those that are not functions making sure that every student gets a chance to throw the ball about the same number of times and several students have the opportunity to try and throw their ball two places at once preventing the relation from being a function. It works the same if the students are numbered

and act as the domain of the relation moving from place to place, when it is not a function the student will not be able to be two places at once.

Step 6: Present the formal definition of a function.

After the game present the formal definition of a function and the traditional way of writing relations to test if they are functions using a few examples.

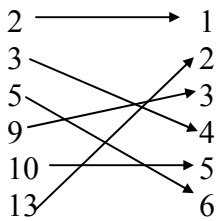
Formal Definition:

A *function* is a relation in which the domain element has only one corresponding range element or a *function* is a relation in which each domain element appears only once.

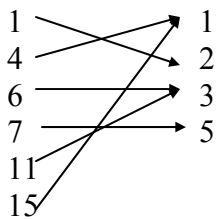
In the broader sense: a *function* is a way to get from a specific input to only one output. Relate the definition back to the ball game.

Traditional Function Test:

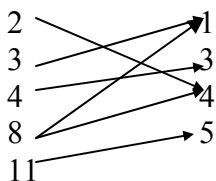
$\{(2,1), (3,4), (5,6), (9,3), (10, 5), (13, 2)\}$ - Function



$\{(1,2), (4,1), (6,3), (7, 5), (11,3), (15, 1)\}$ - Function



$\{(2, 4), (8, 4), (3, 1), (11, 5), (4, 3), (8, 1)\}$ - Not a Function



Activity 2: Function Song (Boyd, 1989). (Another way to remember the definition of a function as it applies to relations)

The purpose of this activity is to give the students another way to remember which variable is the domain and which variable is the range and help them remember the definition of a function.

Materials: A good sense of humor and a patient class.

To the tune of Home on the Range

OH, GIVE ME A SET
OF ORDERED PAIRS, YET
THE X'S MAKE THE DOMAIN.
A FUNCTION IS SEEN
WHEN THE X'S, I MEAN,
ARE DIVERSE AND
THEY CAUSE YOU NO PAIN.

Y'S MAKE UP THE RANGE.
THE RANGE CAN ALL BE THE SAME.
UNLIKE THE X, THEY CAN BE
ALL ALIKE
THE DOMAIN BRINGS THE FUNCTION THE FAME.

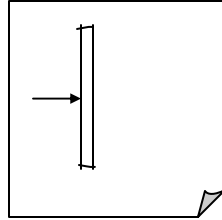
Activity 3: A New Vertical Line Test (Given a relation graphically determine if the relation a function)

The purpose of this activity is to present the vertical line test in a new, perhaps more concrete, way. This is slightly different than the vertical line test that most teachers use. The theory is that this technique will work better than the pencil or side of the paper for the vertical line because the arrow and the limited view help focus the attention on the line. The single arrow reinforces the need for the graph to only cross the vertical line once. This may eliminate some of the confusion.

Materials: each student will need several sheets of graph paper, a piece of computer paper, a pair of scissors (can be shared by several), and a writing implement.

Step 1: Make the vertical line

Have the students cut a thin slit in the center of their computer paper. This becomes the vertical line for the vertical-line test. Have the students draw an arrow pointing to the slit.



Step 2: Write a relation on the overhead.

Write a relation on the overhead that is a function. For example $\{(2,4), (-3,5), (1,7), (3,6)\}$, re-emphasize the different names for the x-coordinate and the y-coordinate, and have the students verify that this relation is a function non-graphically.

Step 3: Graph the relation.

Helpful suggestions to help students remember how to graph points and remember which axis is which:

- Use a baseball analogy. The runner has to be safe (arms crossing body horizontally, x-axis) before the coach can jump up and down excitedly (jump vertically, y-axis).
- Use a bedtime analogy. They have to sleep at night lying down (horizontal, x-axis) before they can wake up and get out of bed (vertical, y-axis).
- Look at the shape of the letters. Point out that y is a taller letter than x so y is the vertical axis and x shorter so x is the horizontal.

Step 4: Test the relation graphically.

Have the students graph all the points. Then moving the computer paper over the graph with the line vertical, have the students make the arrow point to each point as they pass it. If this is possible, then it is a function. Write a relation on the overhead that is not a function. For

example $\{(3,2), (-2,0), (3,-4), (1,5)\}$, have the students graph all the points. Then move their computer paper with the slit vertical across the graph. They should notice that when they get to 3 on the positive x-axis they cannot point at both $(3, 2)$ and $(3, -4)$. Therefore it is not a function. Give the students some more examples of both. Explain the definition of function again and state that the x-coordinate has to know where to go in order for the relation to be a function, in other words each domain element can be paired with only one range element.

Activity 4: A New Vertical Line Test II (Given an equation or a graph test if it a function graphically using the vertical-line test)

The purpose of this activity is to broaden the concept of function past that of a relation and show how the vertical line test is applicable to graphs of equations. There is not an activity for showing how equations can be functions as well as relations, but the concept is a simple transition that should not be difficult to make, especially given that the students should have explored domain and range with graphs and equations already.

Materials: a piece of computer paper for each student (to construct the vertical line), a pair of scissors (can be shared by several), several graphs of functions and several graphs that are not functions with the equations when possible. (Most Algebra text books or work books will already have these graphs, if not the graphs from “Domain and Range Activity 2: Find it!” can be used.)

Step 1: Construct the vertical line as in the previous activity.

Step 2: Either give the students graphs, or write equations and let the students graph them (either with calculator or not).

Step 3: Test the relation graphically.

By moving the computer paper with the line vertical, trace the graph from left to right keeping the arrow on the line. It is all right if the arrow has to jump from one place on the graph to another as long as it can trace the whole graph from left to right. If this is possible, then it is a function.

IV. FUNCTION MANIPULATION

Goal: To help students understand how functions work, manipulate the functions in different ways, and appreciate the usefulness and beauty of functions.

A. Standards:

NCTM Algebra Standards for grades 9-12

Standard 1: Understand Patterns, relations, and functions.

d) Understand and perform transformations such as arithmetically combining, composing and inverting commonly used functions. (National Council of Teachers of Mathematics, 2000, p 395)

B. Background

Included in “Function Manipulation” is plugging-in to functions to find results, finding a function rule if given functional relations, and adding, subtracting, multiplying, and dividing functions. Also encompassed in the computing of operations on functions, the student should understand the similarity between functions and real numbers. It is advantageous that the students understand why these tasks are not only possible but also desirable. Some practical instances where function manipulation is used can further this goal.

Once again the grade function could be useful for teaching practicality. A percent is plugged-in to a function, and out comes a letter grade. This function is a piecewise-defined function. It shows that any percent greater than or equal to 90 but less than or equal to 100 gets a grade of A, any percent greater than or equal to 80 but less than 90 gets a B and so on. Every percent between 0 and 100 is assigned exactly one grade.

$$f(x) = \left\{ \begin{array}{l} A \ni 90 \leq x \leq 100 \\ B \ni 80 \leq x < 90 \\ C \ni 70 \leq x < 80 \\ D \ni 60 \leq x < 70 \\ F \ni 0 \leq x < 60 \end{array} \right\}$$

Another example could be the scale function. If artist is trying to make a scale drawing of a landscape he or she could plug in the length of the field in meters into a scale function and get inches. $f(x) = 39.37x$, where x is in meters and $f(x)$ is in inches.

To emphasize the operations on functions the grade example can be used once again. If a teacher was trying to calculate grades, she may need to add more than one grade function or divide the total number of grades taken. If a runner were attempting to calculate his average speed for the mile, he would have to first use a function to calculate his velocity per lap and then divide by the number of laps he ran. If a student was trying to make enough cookies for their entire class and needed three times as many cookies as the recipe produced, he or she would have to multiply the recipe function (all of the ingredient amounts) by the constant function $f(x) = 3$ to get three times the batter. Some of these may help students to see the relationship between functions and real numbers. In fact students can be taught to think of real numbers as constant functions and then every operation becomes a manipulation of functions. This does not make basic operations more difficult, it only makes functions seem easier.

One major reason functions are so difficult for students, and even adults, to understand is lack of instruction or emphasis on proper notation (Harel & Dubinsky, 1992). The function notation is set up to explain exactly how a function works. After students understand what a function is and how it is useful they should learn what the function notation means, with emphasis on the significance and the logic behind the notation (Harel & Dubinsky, 1992).

A function does something. It changes something to make something else. It transforms reality in an organized way. Think of the variable in a function as a hole. A hole needs to be filled. In the function rule $f(x) = 3x + 2$ for example the “f” is the name of the function (although it could be “g” or “h” or “y”). The “x’s” are the holes, the places where different elements, be they other variables, other functions, or numbers, go. A variable is called a variable because it varies, or changes. The $3(\) + 2$ describes what happens to the elements plugged into the holes. For example, if a student wants to find out what would happen to the number **2**, what they are really looking for is $f(\mathbf{2})$ “f of 2”. They plug **2** into the hole and the equation become $f(\mathbf{2}) = 3(\mathbf{2}) + 2$. Meaning that first **2** is multiplied by 3 then 2 is added to the product. The total “f(whatever)”, in this case $f(\mathbf{2}) = 8$ is the final result. The actions of the function are completely

explained by the notation. If a student wanted to know what would happen to $h+4$, what they are really looking for is $f(h+4)$ “f of $h+4$ ”. The first step is to plug $h+4$ into the hole made by the x . The equation then becomes $f(h+4) = 3(h+4) + 2$. This means that first the 3 is distributed through the parenthesis, then like terms are combined and the equation is simplified. The final result is $f(h+4) = 3h + 14$. Understanding that the variable x is a hole to be filled makes the function notation clearer (Harel & Dubinsky, 1992).

Activity 1: Fun Plugging into Functions (If given a number or an expression that is in the domain of a function find the corresponding range element algebraically, and understand the function notation)

This activity helps emphasize the idea of the variable in a function being a “hole” to fill with something else. By making actual holes in the function where the variable would be, this idea, and the proper use of function notation is hopefully cemented. This activity will greatly help the visual and tactile learners because of the physical hole created and manually filled, but the verbal learners will benefit by being able to read the function both before the variable is taken out (making a hole) and after a new variable or number is put in its place (filling the hole). By working both individually and in groups the students will take personal responsibility for their understanding but be willing to ask for help when they need it, and checking their answers to give them a feeling of success (Glasser, 1988). This process should help cement the idea of taking out the variable and replacing it with a different domain element.

Materials: A set of laminated *function cards* and answer sheets one possible set is included in the Appendix titled “Fun Plugging into Functions”

Step 1: Prepare materials

The function cards should have functions with squares cut out in the areas where the variables belong; the answer sheet should have squares positioned in the same places so that when the “function card” is laid on top of the first set of squares in the answer sheet the function will have X’s in the proper place. The other numbers on the answer sheet can be filled in at the teacher’s discretion, or be left empty and given to the students on the overhead in the form of “find f(4).” Sets should be created with the squares in the same place so that the answer sheets can be the same regardless of the function card. The teacher might make more than one set of cards and have different answer sheets for the different sets. Numbering the answer sheets could be helpful in keeping organized, and keys for the different cards could be easily constructed. Other sets could be made if the one in the appendix is not sufficient.

$F(\square) = 2(\square) + 4 = \square$	1
$G(\square) = \frac{2}{3}(\square) - 2 = \square$	2
$H(\square) = -5(\square) + 3 = \square$	3

X	X	□
1	□	□
-3	□	□
1/2	□	□
4	□	□
0	□	□
Function _____		

Step 2: Distribute a function card and a few answer sheets to each student and give directions.

The students should lay the card on an answer sheet lining up the boxes and complete the answer sheet by moving the function card down each row of boxes solving the function with the given domain values. In the set in the appendix the first column should be given to the students, the second column should be the same as the first but filled in by the students as they go, and the last row should be the answer after the computation.

Step 3: Switch Cards

When the students have completed a function card, have them switch with another student. Let them compare answers with other students who have done the same card, and help each other with understanding.

Step 4: (optional) Expand the lesson

For a more advanced lesson, put other domain elements in the initial boxes besides numbers such as “ $x+3$ ” or “ $2h-9$ ” this will help prepare the students for composition of functions.

Step 5: Present the traditional method

It is important to practice this process the traditional way as well, with pencil and paper, either immediately after or the day after so that the students can take the concept and apply it without the cards. This will make it easier to expand the concept.

Activity 2: Rules Rules (If given a relation that is a function, find the function rule that describes the relation.)

The purpose of this activity is to help the students be able to recognize patterns and further the connection between relations as functions and equations as functions. When an equation is a function it is often called a “function rule” because it is a rule i.e. strict set of instructions to get from one value/ object/ etc. to another.

Materials: Pencil and paper for students.

Step 1: Divide your students into groups of three and assign jobs.

Assign different jobs to each member of a group. The first job is to come up with the domain element “the D maker”, the second job is to perform the operations “the function worker”, and the third job is to be the range and the recorder the “rule writer”. Explain the three jobs represent the three parts of a function, the domain element or the starting point, the function

or the operations, and the range element or the final solution. The jobs will rotate throughout the lesson so each person has a chance at each job.

Step 2: Assign functions to groups

Ask the members to all send “the function worker” up to you to talk to. Assign “the function worker” a function and tell them to keep it a secret. Some simple examples are:

$$f(x) = x + 7$$

$$f(x) = x - 3$$

$$f(x) = 4x$$

$$f(x) = x/2$$

The examples should have only one operation to start so that they are easy to figure out, and as the members rotate jobs the functions can get more complicated.

Step 3: Explain the game.

First “the D maker” comes up with a number and tells their group what it is; “the rule writer” records it in a table. *Use an integer just to keep things simple, later remind the students that in most functions the domain could be any real number but they should already know this.*

x	f(x)

Second “the function worker” changes the number by performing the operation and tells the new number to the group and “the rule writer” records this number next to the first in the table. *If “the function worker” needs to write the functions and the numbers down to do the operations, it is perfectly all right... or it could be required that they write the function down and replace the x with each number as long as it is a secret from the rest of the group.*

Third the process is repeated as many times as it takes for “the rule writer” to figure out what is being done to get from the domain to the range. Then “the rule writer” writes down the function

rule and “the function worker” tells him or her if he or she is correct. *If the domain person comes up with the function rule first, they are not allowed to say but they can help.*

x	f(x)
2	8
4	16
3	12
-1	-4

$$f(x) = 4x$$

Fourth the students in the group switch jobs and send the new “function worker” to get another function.

Step 4: Collect the work

The group paper should have all three names and be organized with a table and a rule for each example. It is important to monitor the groups while they are working, but collecting the papers and reviewing gives concrete feedback as to how well the groups understood.

Activity 3: Shapes and Colors (If given two functions, be able to add or subtract, them in any given order with any specified domain element.)

This activity will benefit the visual and the tactile learners. The use of color and shape will help emphasize the importance of *like terms* and help the students remember to check for like terms when simplifying and combine only like terms. The comparison to real numbers will help build on previous schema (Wolfe, 2001) and hopefully make the concept more fully retained and comprehended as a result.

Materials: lots of colored pencils or different colored high lighters and student pencil and paper.

Step 1: Write two functions on the overhead and begin explanation with addition.

For example $f(x) = x^2 + 2x - 3$ and $g(x) = 4x - 1$ to find $f(x) + g(x)$ or $(f+g)(x)$, teach both notations. Write the two functions on the overhead with in parenthesis with a plus sign between.

$$(x^2 + 2x - 3) + (4x - 2)$$

Step 2: Discuss the distributive property

Explain that “+1” is distributed through both sets of parenthesis to get rid of them. The result is: $x^2 + 2x - 3 + 4x - 2$

Step 3: Color and/ or draw shapes

Color “like terms” the same color. For example color x^2 red and/or circle, $2x$ and $4x$ blue and/or triangle and -3 and -2 yellow and/or square. If colors are not available draw different shapes around the like terms or do both. This is good because it emphasizes like terms, is very visual, and when the students color or draw shapes on their own, it helps the tactile students and forces all students to be aware of the importance of like terms and emphasize that only like terms can be combined

$$\textcircled{x^2} + \triangle 2x - \square 3 + \triangle 4x - \square 2$$

Step 4: Rearrange the equation

Rewrite the equation with the same colors next to each other and recolor if desired. Take this opportunity to discuss the commutative property of addition. Then help the students combine like terms.

$$x^2 + 2x + 4x - 3 - 2 = x^2 + 6x - 5$$

Step 5: More

Ask the students to find $g(x) + f(x)$ or $(g+f)(x)$. This should further emphasize the similarity between real numbers and functions.

Step 6: Subtraction

This same process can be used for subtracting functions. $f(x) - g(x)$ or $(f-g)(x)$. Where this time “+1” will be distributed through the first set of parentheses and “-1” through the second to drop the parentheses.

Step 7: Expand

Then teach the students how to solve equations like $2f(x) - 3g(x)$ or $(2f-3g)(x)$, etc by following the same process.

Activity 4: Shapes and Colors II (If given two functions, be able to multiply them in any given order with any specified domain element.)

This activity will build multiplication of functions step by step, emphasize like terms, show the double distributive property as it relates to functions, and demonstrate once again how functions are similar to real numbers.

A note on division of functions: at this stage it is more important that students know the correct form in which to write the equations, which should be shown step-by-step with simpler functions first. It is not necessary that the students fully simplify the solution, nor that they learn long division or know factoring of polynomials at this stage. Students should write the division as fractions and with a straight bar instead of a slanted bar. This will help them later when it comes to factoring with canceling and long division.

Materials: Same as in previous activity.

Step 1: Write two functions on the overhead.

Start by giving a variable function and a constant function. For example $f(x) = x^2 + 2x - 3$, and $g(x) = -1$ to find $g(x)f(x)$ or $(g \cdot f)(x)$ means find $(-1)(x^2 + 2x - 3)$ write the parenthesis around both with nothing in between so it is clear it is multiplication.

Step 2: Perform the multiplication

This should be easy because it is using the distributive property, which they just reviewed. Do the same process with coloring “like terms,” and reemphasize that only like terms can be combined.

$$\textcircled{x^2} \textcircled{-2x} \textcircled{+3}$$

Then perform the operation $f(x)g(x)$ or $(f \cdot g)(x)$ to get the same result.

Step 3: Build $g(x)$ up a step and perform multiplication again.

Next make $g(x)$ a little more complex, for example $f(x) = x^2 + 2x - 3$, and $g(x) = x$.

Go through the same process to find $g(x)f(x)$ showing that the distributive property works the same, color “like terms” and combine: $(x)(x^2 + 2x - 3) = x^3 + 2x^2 - 3x$

Step 4: Build $g(x)$ up another step and perform multiplication again.

Next make $g(x)$ even more complex, $f(x) = x^2 + 2x - 3$, and $g(x) = 4x$ and follow the same process and color like terms. $g(x)f(x) = (4x)(x^2 + 2x - 3) = 4x^3 + 8x^2 - 12x$.

Step 5: Build $g(x)$ up one more time.

Next use the functions $f(x) = x^2 + 2x - 3$, and $g(x) = 4x - 1$. Notice that $g(x)$ was built from its simplest components. This building is important so the students see simplicity and practice drawing connections. Find $g(x)f(x)$ or $(g \cdot f)(x)$. $(4x-1)(x^2 + 2x - 3)$

Step 6: Perform multiplication.

Point out to the students that they have already done both parts of the problem.

$(4x-1)(x^2 + 2x - 3)$ is exactly the same as $(4x)(x^2 + 2x - 3) + (-1)(x^2 + 2x - 3)$. This is called the double distributive method of multiplying functions because you have to distribute twice: $4x^3 + 8x^2 - 12x + -x^2 - 2x + 3$

Color or mark “like terms”

$$\textcircled{4x^3} + \triangle 8x^2 - \square 12x + \triangle -x^2 - \square 2x + \star 3$$

Then combine: $4x^3 + 7x^2 - 14x + 3$.

Step 7: More

Perform $f(x)g(x)$ or $(f \cdot g)(x)$ and show that $(x^2 + 2x - 3)(4x-1)$ is the same as $(x^2)(4x-1) + (2x)(4x-1) + (-3)(4x-1)$, distribute, color, rearrange, and combine.

$$= 4x^3 + 7x^2 - 14x + 3.$$

Follow up Activity: Combining skills

The students should already know how to plug in domain values into functions; so that they do not lose this skill, and realize that the skill is applicable, give the students some problems where they are asked them to find $(f \cdot g)(3)$ for example or $(f + g)(x + 2)$. Have them first do the operation and then substitute in the domain element, or substitute in the domain element then do the operation.

Activity 5: If Given the Domain, Find the Range (If given a number find the corresponding range element graphically)

The purpose of the activity is to have partial graphs where the students can find the x value on the axis, follow it vertically to the function then horizontally to the y -axis to produce an answer; then be able to calculate a range value algebraically, and finally check their answer by continuing the graph.

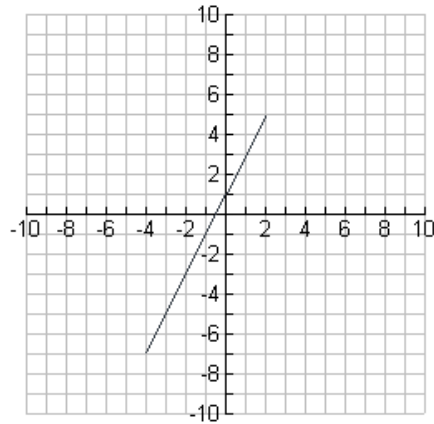
Materials: A sheet of different partial graphs for the students to use and areas for questions and answers corresponding to those graphs. A possible sheet and the corresponding answers is included in the appendix titled “If Given the Domain, Find the Range”

Step 1: Review

The students should already realize that functions can be represented graphically. Still, repetition helps aid retention (National Council of Teachers of Mathematics, 2000).

Step 2: Demonstrate finding range from the domain graphically.

Use the fact that functions can be represented graphically to find the range when given the domain. Start with a simple example like $f(x) = 2x + 1$. A partial graph looks like:



The students could be asked to find $f(-2)$, which means that they go over -2 on the x-axis and go vertically, in this case down, to find where the line intersects -2 . Once the graph is reached go horizontally, in this case right, back to the y-axis to find the corresponding value, which is -3 . So $f(-2) = -3$. Then they could be asked to find $f(-2)$ algebraically. $f(-2) = 2(-2) + 1 = -3$. The students could also be asked to find $f(4)$ algebraically, $f(4) = 2(4) + 1 = 9$, graph the point $(4, 9)$, and verify that if they continued the line this point would work.

Step 3: Distribute the “If Given the Domain, Find the Range” activity sheets, and allow the students to work in pairs (Glasser, 1998).

The questions should help students see patterns about the graphs as well as help them see the relationship between the graphical interpretation and the algebraic interpretation. This idea of relating the picture of the graph to the algebra is extremely important to the idea behind functions and a good way to help more than just the visual learners (Harel & Dubinsky, 1992). The questions should help the students gain a better understanding of why and when graphs are useful. The written answers at the end help the tactile and visual learners cement the concepts they have learned, and the auditory learners will be helped by the discussion of the questions.

V. FUNCTION COMPOSITION AND INVERSES

Goal: To help students understand composition of functions, inverses, their uses.

A. Standards

NCTM Algebra Standards for grades 9-12

Standard 1: Understand Patterns, relations, and functions.

d) Understand and perform transformations such as arithmetically combining, composing and inverting commonly used functions. (National Council of Teachers of Mathematics, 2000, p 395)

B. Background

Function composition is critical to learn because in many cases the basic operations (adding, subtracting, multiplying, and dividing) are not enough to get the desired solution. The theory behind composition of functions is that there are times when one wants to perform one function first, and then use that answer in another function to get a final solution. In these cases the middle solution usually is not even important to get an answer for except in that it affects the final solution. Learning function composition can simplify the process of finding solutions to problems or answers to questions so that one does not have to make so many individual calculations. It is not only simpler but also more accurate because then one does not have to keep track of so many numbers or worry about significant figures or keeping enough decimal places until the end of the problem (Harel & Dubinsky, 1992).

The basic concept behind composition of functions is the same as that of solving normal functions for an expression or a number. The important thing to remember is that the variable is still a “hole” that needs to be filled. In the case of composition of functions the “hole” is filled with another function instead of a number or an expression, but one must remember that a function is just a number or expression with certain properties. The concept is the same. Once again, a clear understanding of function notation will help with the calculations or

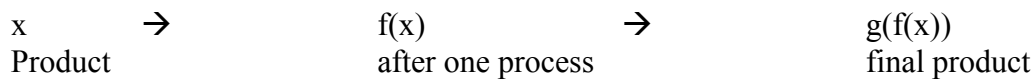
simplifications. If the student understands this, function composition will come much more easily, make more sense, and hopefully be better retained.

Some real life times when composition of functions is used, even if those using it do not know what it is they are really doing include anything where parts are easier to understand than the whole, but the knowing the exact answers for the parts are unimportant:

Composition can be thought of as something like plumbing or electrical wiring, where we buy parts off the shelf and connect them end-to-end to make a new device or to wire a house or factory according to our needs. We know how each pipe, wire, switch, etc. functions (pun intended), and we know how they combine, so we can understand the whole complicated system in terms of its parts. Without the function concept, we couldn't do that in math... The concept of a function is also central to computer programming, though the details are somewhat different there. Most of what a programmer writes consists of 'functions' that do parts of the work of the program. By designing functions that do little pieces, we can string them together to do more complicated things without looking so complicated... [thus] we are able to write our own functions, and then put those together to make larger functions (Ask Dr. Math, 2005).

Some simple ways that composition of functions can be demonstrated or explained to students could be to relate it to a process. A raw product is the input, like crude oil or cotton plant, and we first need to change it or refine it to obtain a secondary product gasoline or cotton. This secondary product is then further refined and processed obtaining the desired product such as energy for a car to move or cloths to wear.

This could be represented functionally by:

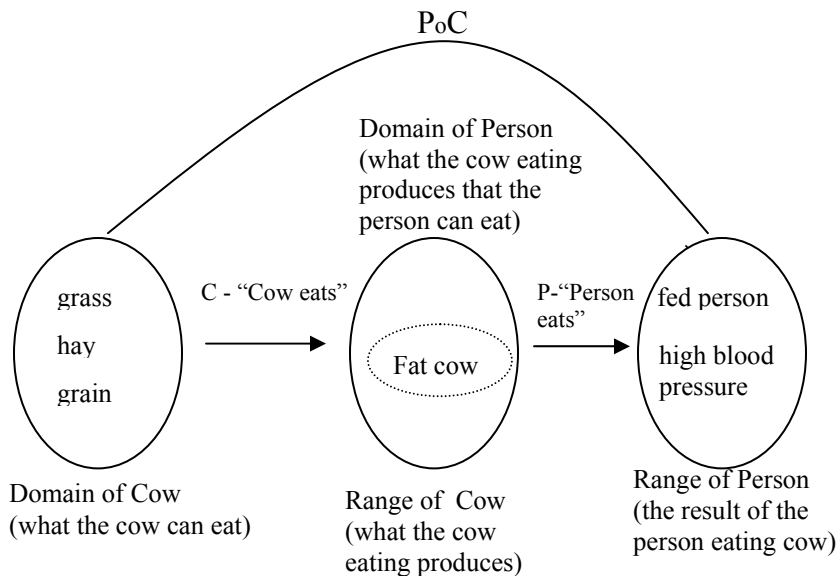


Another example of function composition, or a way to think about function composition is the food chain. Take grass (g) as the first input; then the cow (c), being a function, "cow eats" the grass. Next, here comes a third animal; for instance the person (p), and the function "person eats" the cow. The best way to denote this is: p(c(g)). The parentheses denote the walls of the stomachs of those animals, though slightly crude as an example, it makes sense (Ask Dr. Math, 2005).

Many of the functions students have already seen and worked with can be broken into parts and written as a composition of functions. For example $f(x) = 2x + 3$. Let $g(x) = 2x$ and let $h(x) = x + 3$, then the composition of the two functions g and h is $(h \circ g)(x) = h(g(x)) = h(2x) = 2x + 3 = f(x)$. Composition of functions is the process of taking simple functions, or simple processes, and building more complicated ones.

Composition of functions can also be shown pictorially. This could emphasize the concepts of domain and range and how they relate to composition of functions. Students learn different ways, and many visual learners need a picture of what is happening; the commonly known pictorial method of functions is worth analyzing and explaining to the students.

One can also use the food chain example to think about domain and range. The domain for the cow function consists of things that the “cow eats,” such as grass or hay. The fattened cow (i.e. the range of the cow function, or what happened when the cow eats) then becomes the domain for the person function, meaning the person can eat the cow. Though the person can eat many other things besides the cow, things like chicken and pork are not in the range of the cow (i.e. the result from the cow eating the grass). Also, it would make no sense to have the function $p(g)$ because people do not eat grass showing that grass is not in the domain of the person function. Here is a visual of the function composition just explained:



The other part of this group of activities has to do with inverses. A common inverse function used in math that the students at this level may be familiar with is the square root. It is an inverse because it undoes the square function. Other inverse functions they may or may not be familiar with include the arc functions in trigonometry and the log function to undo an exponential.

Activity 1: Function Composition (Given two functions, be able to compose the functions in any given order algebraically and find the composition of the function at some domain element)

The purpose of this activity is to make comprehension of composition of functions more accessible by organizing the process in a chart with clear steps and building on the previous schema built when performing function rules.

Materials: Pencil and paper for the students and chart with the steps. A step-by-step activity sheet is included in the appendix titled “Function Composition.”

Step 1: Review how function rules work.

Write a function on the overhead, for example $f(x) = 2x - 3$. And remind the students how to find $f(4)$, i.e. take out the “x” and replace it with “4”: $f(4) = 2(4) - 3 = 5$.

Step 2: Perform the function rule with a different variable

Now ask the students to find $f(p)$, i.e. take out the “x” and replace it with “p”: $f(p) = 2p - 3$.

Step 3: Now teach the composition

For example using the functions: $f(x) = 2x - 3$ and $g(x) = 4x - 1$.

Give the students a set of steps for example to find $(f \circ g)(2)$: Start by finding $(f \circ g)(x)$ then finding $(f \circ g)(2)$.

First: rewrite $(f \circ g)(x)$ as $(f(g(x)))$

$$(f \circ g)(x) = f(g(x))$$

This will emphasize that f is the function being filled, and g is what is filling the function f . The function g still has a hole in it, the x . Tell the students that this hole will be filled later with other things, but make sure they know it is still a hole.

Second: drop the x in $g(x)$ for simplification.

$$f(g(x)) = f(g)$$

This simplifies the function so that g is easier to plug into f . But having the function written the other way first emphasizes that g is not a variable, g is a function.

Third: Fill the hole in function f with g .

$$f(g) = 2g - 3$$

This shows the hole being filled in properly with the function g . Because they have just practiced this process, this should be easy for the students to see and make the replacement correctly.

Forth: replace g with the function $g(x)$

$$f(g(x)) = 2(4x - 1) - 3$$

This is the step where the composition actually takes place. It should be easy once the g is in place to simply take the g out and put in the function.

Fifth: distribute and simplify

$$f(g(x)) = 8x - 2 - 3 = 8x - 5$$

This is the easy part. The students should feel comfortable when they get to this step.

Sixth: plug in the value if there is one to solve for, and simplify.

$$f(g(2)) = 8(2) - 5 = 11$$

It helps to line everything up as one proceeds through the steps so that the students are clear on what is being replaced. When needed, draw arrows from the previous step to clarify. Teach the students to write going down the page as well, lining up their work. If the students practice writing the problems the way the teacher does they will have a clearer understanding of the notation, and be able to follow the examples given them better (Harel & Dubinsky, 1992).

Step 4: distribute a chart with ordered steps and let the students practice.

Let the students practice some of these other compositions either with the same equations or without using a chart to organize their information and remind them of the steps.

The students should now be able to find any combination of compositions and solve the compositions with a specified domain element. The possible compositions with two equations are $(f \circ g)(x)$, $(f \circ f)(x)$, $(g \circ f)(x)$, and $(g \circ g)(x)$.

Activity 2: Pictures of Composition (Given two functions, be able to compose the functions in any given order pictorially)

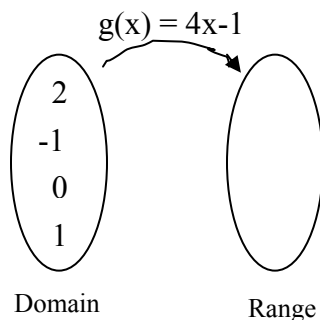
The purpose of this activity is to review the concepts of domain and range and understand what is happening to the domain and range when functions are composed. The pictorial method is especially helpful in understanding that another definition of compound functions is where the range of one function becomes the domain of the function with which it is being composed.

Materials: Pencils and paper and patty paper (translucent paper that can be placed on objects and traced, similar to wax paper and if not available wax paper will work) for the students.

Step 1: Pick functions and show a single function with pictures.

Pick two simple functions, for example $g(x) = 4x-1$ and $f(x) = 2x-3$. Guide the students to draw a circle for the domain of g and the range of g , and label the circles. Talk about what goes in each circle and pick a few specific values in the domain of g to calculate solutions.

When drawing the picture, the over arching arrow joining the two circles represents the function performing a process or action on the numbers or objects in one circle to produce the numbers or objects in the next. It is important to have the arching arrow to show that not every number that could be plugged in has been. Write the chosen domain elements in the domain circle.

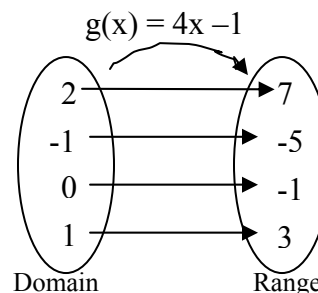


Step 2: Use a wide table to calculate the range values.

This is beneficial because it gives the students more practice on plugging into functions.

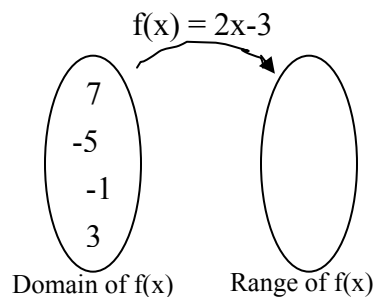
Once the range values are calculated, write them in the range circle and draw arrows connecting the appropriate values.

$g(x)$	$=4x-1$	$=$
2	$4(2)-1$	7
-1	$4(-1)-1$	-5
0	$4(0)-1$	-1
1	$4(1)-1$	3



Step 3: The using the patty paper, have the students trace the range of g .

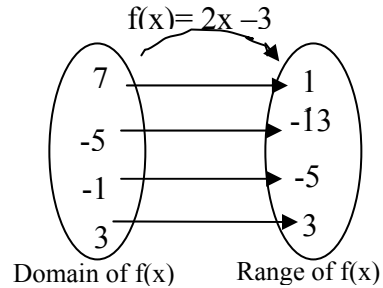
The range of g will become the domain of f . After the students trace the range of g (circle and numbers) have them re-label the range as “domain of $f(x)$ ” and write on their previous paper “of $g(x)$ ” next to the range of g . Then draw an empty circle for the range of f and label it as such.



Step 4: Use a wide table to calculate the range values.

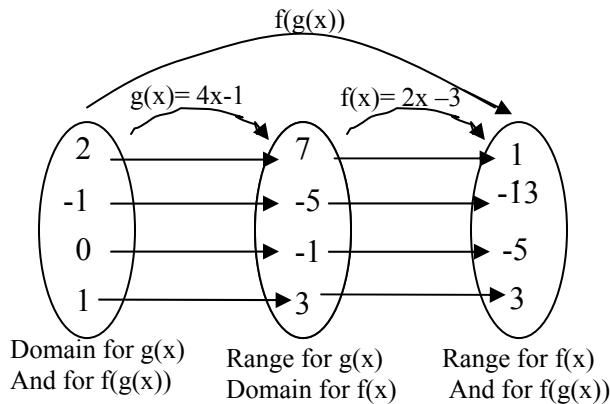
Since the range of g has become the domain of f , now new range values are calculated with the new domain. They are then written in the range circle arrows drawn to connect the appropriate values.

$f(x)$	$=2x-3$	$=$
7	$2(7)-3$	11
-5	$2(-5)-3$	-13
-1	$2(-1)-3$	-5
3	$2(3)-3$	3



Step 5: Lay the patty paper back on the original paper and finish labels.

The papers should line up with the range circle of g and the domain circle of f overlapping. Draw an overarching arrow from the domain of g all the way over to the range of f . This arrow represents $f(g(x))$. Label as such and make the labels on the circles more specific.



Step 6: Make a very wide table to show the composition with the numbers.

Composition can be shown both ways, either by finding $g(x)$ first and then plugging the specific range values for $g(x)$ in for x in f , or g can be plugged in for x in f and then plug in the domain values. The answer is the same.

x	$g(x) = 4x-1$	g	$f(g)=2g-3$	$f(g)$
2	$4(2)-1$	7	$2(7)-3$	11
-1	$4(-1)-1$	-5	$2(-5)-3$	-13
0	$4(0)-1$	-1	$2(-1)-3$	-5
1	$4(1)-1$	3	$2(3)-3$	3

x	$f(g(x))=2(4x-1)-3$ $= 8x-5$	$f(g)$
2	$8(2)-5$	11
-1	$8(-1)-5$	-13
0	$8(0)-5$	-5
1	$8(1)-5$	3

Step 7: Explain new understanding of composition of functions.

Using the patty paper the students can see both functions together; this helps show that g fits inside f . The process of using the range of $g(x)$ for the domain of $f(x)$ sticks the two functions together. This makes a compound function, hence the operation “function composition”. The total picture seems to go backwards according to the name, but because functions, like real numbers, work from the inside out following order of operations, in the picture of the function $f(g(x))$ the function $g(x)$ has to come first.

Activity 3: Finding the Function Inverse (If given a function, algebraically find the inverse of the function)

The purpose of this activity is to find the inverse of a function algebraically as well as understand the definition and the purpose of inverses. This activity also teaches how to check that the inverse found algebraically is correct.

Materials: Pencil and paper for the students to make the tables while going through the notes and “Function Inverse” step-by-step guidance sheet included in the appendix.

Step 1: Have a discussion about what inverses are, and the purpose of inverses.

Here are some well-worded explanations that are applicable to high school students.

Whenever they undo something that they or someone else did, they use an inverse function, whether it's untying a knot or solving a puzzle or decoding a secret message. When a computer reads a number you type in and converts it to binary for internal storage, then prints it out again on the screen for you to see, it's doing an inverse function. [For an applicable example] How about this: When someone calls you on the phone, he or she looks up your number in a phone book (a function from names to phone numbers). When Caller ID shows who is calling, it has performed the inverse function, finding the name corresponding to the number (Ask Dr. Math, 2005).

Inverse functions are important because sometimes in life you are given a puzzle, or a final product and it is important to figure out how you got it. For example say you go to the

grocery store and you spend \$100. Using the inverse function, in other words the inverse of adding all the prices of the individual groceries, would be splitting the price from one total price to the prices of the individual groceries. If you are at a fast food restaurant with your friend and he pays \$10 for both your foods in the drive through, you use an inverse function to figure out if you owe him \$4 or \$6. If you have a toaster that breaks and you want to figure out what is the matter with it, you take it apart using an inverse function and, by keeping track of how you took it apart, you can use the inverse of the inverse and put it back together.

Step 2: introduce the students to some really simple examples of inverse functions.

Consider the function $f(x) = 2x$. The inverse (opposite) of multiplication is division.

This means that division undoes multiplication. Students usually learn about inverse operations in solving equations. This usually comes in Algebra curriculum before functions. So the students should have a clear understanding of inverse operations. The task is then to broaden their understanding of inverse operations to inverse functions. Ask the students what they think the inverse function should be, and teach the notation for inverse function, $f^{-1}(x)$, and read “f inverse of x.” The expected inverse for $f(x) = 2x$ would seem to be $f^{-1}(x) = x \div 2$, or

$$f^{-1}(x) = \frac{1}{2}x = \frac{x}{2}.$$

Step 2: Check the inverse using a table.

Start by finding $f(x)$ for a few values, then see if $f^{-1}(x)$ undoes $f(x)$. In other words plug the range elements of f in for x in the alleged inverse to see if you get the values you started with... if this happens the inverse function has been successfully found.

x	f(x)=2x	f(x)
-1	-1(2)	-2
0	0(2)	0
2	2(2)	4
3	3(2)	6

\longrightarrow

f(x)	$f^{-1}(f(x)) = \frac{x}{2}$	$f^{-1}(f(x))$
-2	-2/2	-1
0	0/2	0
4	4/2	2
6	6/2	3

Notice that the column for $f^{-1}(f(x))$ is equal to the column for x . Like it is traditionally taught if $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ then they are inverses. In simple words, the functions undo each other and you get what you started with (the domain or x) they are inverse functions.

Step 3: More simple examples, and table similarities emphasized.

Find the inverse of $f(x) = x + 3$, so inverse (opposite) of addition is subtraction.

Therefore $f^{-1}(x) = x - 3$. Then show how the columns switch places. The first column in the first table became the last column in the second table, and the last column in the first table became the first column in the second table. The x became $f(x)$ [or y] and $f(x)$ [or y] became x .

Step 4: Teach the process to finding the inverse with steps.

Here are the steps with two examples:

Functions:

$$f(x) = x + 3 \qquad g(x) = \frac{3x + 2}{x - 1}$$

First: replace $f(x)$ and $g(x)$ with a single variable for simplicity, it could be y as is traditionally used or just f and g . Either way the students need to understand that they have not changed the problem by making this substitution. They should also understand that the variable, whether it is “ f ” “ g ” or “ y ” still represents the range of the function.

$$f = x + 3 \qquad g = \frac{3x + 2}{x - 1}$$

Second: Switch the domain and range variables by switching f and x and switching g and x . This should be compared to the tables when the columns switch. Remind the students that the inverse is going backwards, or undoing the function. Therefore what was once the range becomes the domain and what was once the domain becomes the range, hence the switching of the variables. It is important to explain the reason behind this step to the students. To just tell them to switch without an explanation cheats them out of the understanding of functions and

inverses that could be gained (Glasser, 1988). It also makes it more difficult for them to remember what to do if they do not know why (Glasser, 1988).

$$x = f + 3$$

$$x = \frac{3g + 2}{g - 1}$$

Third: Solve for the new domain element, the new f and the new g. This step is important because a function if it is a function, it is always written with the dependent variable alone on one side so that it can be more easily calculated. Furthermore, the inverse equation cannot be manipulated or used as a function unless it is in the proper form.

$$x = f + 3$$

$$\begin{array}{r} -3 \quad -3 \\ \hline \end{array}$$

$$x - 3 = f$$

$$f = x - 3$$

Subtract 3 from both sides

$$x = \frac{3g + 2}{g - 1}$$

$$(g-1)x = \frac{3g + 2}{g - 1}(g - 1)$$

$$(g-1)x = 3g + 2$$

$$\begin{array}{r} gx - x = 3g + 2 \\ +x \quad +x \\ \hline \end{array}$$

$$\begin{array}{r} gx = 3g + 2 + x \\ -3g \quad -3g \\ \hline \end{array}$$

$$gx - 3g = 2 + x$$

$$g(x-3) = 2 + x$$

$$g = \frac{x + 2}{x - 3}$$

Multiply by the denominator to cancel on one side

Distribute

X's on one side g's on the other

Factor out g.

Divide to solve for g.

Fourth: Rename each equation with the proper title to show that it is the inverse of the function.

$$f^{-1}(x) = x - 3$$

$$g^{-1}(x) = \frac{x + 2}{x - 3}$$

Fifth: Check the solutions.

Step 4: Distribute “Finding the Function Inverse” and let the students practice by filling in the rest of the chart.

Activity 4: Functions and their Inverses (Discover what it means to have an inverse graphically and test if the inverse is itself a function)

The purpose of this activity is to discover that inverse functions reflect across the line $y = x$ because this is the only place where when x and y are switched (as when happens to find the inverse) everything on the line stays the same. The horizontal line test can be discussed simultaneously to see if the inverse is a function, for every function corresponds to an inverse relation; the question is whether or not the inverse relation is itself a function.

Materials: “Functions and their Inverses” which is a set of graphs of functions and their inverses, functions without their inverse functions, and functions without inverses (available in the appendix), and a mira for each student (a math tool that shows a reflection but also allows the students to see both sides).

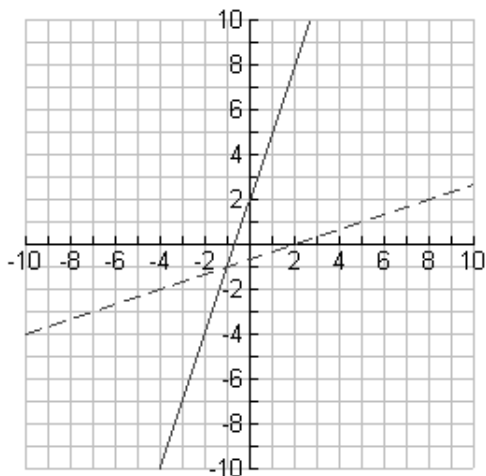
Step 1: Experiment with the mira.

Distribute a set of graphs and their inverses and a mira to each student. Ask the students to use the graphs and the mira and see if they can figure out where to place the mira to get the graphs to lay one on the other. Ask them if this works for all the examples you have given them. Have them try to figure out what the equation of the line they have to place the mira on for this to happen. They should come up with the line $y = x$. The students may need to be reminded of equations in $y = mx + b$ form and how to graph them before this is clear.

Step 2: Discuss what is happening.

The x 's and y 's are switching. Pick an easily distinguishable point on one of the graphs, for instance the point $(0,2)$ on the graph below. The inverse of the point $(2,0)$ lies on the inverse

of the graph directly reflected across the line $y = x$. The point $(-1, -1)$ lies on both the graph and the inverse because the x and the y are the same.



Step 3: Give the students some more graphs of functions without the accompanying inverses.

With this new set of graphs see if the students can use the mira on the appropriate line ($y = x$) to trace the inverse.

Step 4: Teach the horizontal line test.

Give the students some graphs of functions that do not have inverses. Have the students place the mira in the appropriate place and trace the graph of the inverse. Ask the students if the inverse they just traced is a function by using the vertical line test. When they discover one that is not have them make a guess as to why. Then go back to the original graph and demonstrate how the horizontal line test fails. Go back to a previous example where the inverse is a function. Ask them to check if the inverse is a function using the vertical line test on the inverse, and then show how the original function passes the vertical line test.

VI. FAMILIES OF FUNCTIONS

Goal: Students will understand the different families of functions and how they relate to each other.

A. Standards

NCTM Algebra Standards for grades 9-12

Standard 1: Understand Patterns, relations, and functions.

- a) Generalize patterns using explicitly defined and recursively-defined functions.
- c) Analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior.
- e) Understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions.
- f) Interpret representations of functions of two variables.

Standard 2: Represent and analyze mathematical situations and structures using algebraic symbols.

- a) Understand the meaning of equivalent forms of expressions, equations, inequalities, and relations
- b) Write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency- mentally or with paper and pencil in simple cases and using technology in all cases.
- c) Use symbolic algebra to represent and explain mathematical relationships
- d) Use a variety of symbolic representations, including recursive and parametric equations, for functions and relations
- e) Judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.

Standard 3: Use mathematical models to represent and understand quantitative relationships

- d) Identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships (National Council of Teachers of Mathematics, 2000, p 395).

B. Background

Activity 1: Building Connections (Burk, 2000)

The purpose of this activity is to help students understand the connections between linear functions and higher order polynomial functions and learn how to build one from the other.

Many students learn about the “root” of polynomial functions but few teacher or students take the time to understand how the polynomial function is actually built from the roots graphically.

Graphing calculators can be used to help the students see the algebraic representation of the functions alongside the graphic representations as the polynomial functions are built from their linear roots.

Materials: “Building Polynomial Functions” found in the appendix, and graphing calculators can also be used to supplement these sheets.

Step 1: Distribute the sheets to the students.

The sheets are labeled and can be done in one to three lessons. 1a and 1b go together; sheet 2 and 3 can be given at the same time as 1 or as two different lessons. The sheets are self-explanatory and the reading individually is part of the learning gained from this activity.

Background and Explanation:

A major thrust of Principles and Standards for School Mathematics is that teachers should help students see mathematics as an integrated whole rather than as a series of isolated topics. "Mathematics makes more sense and is easier to remember and apply when students can connect new knowledge to existing knowledge in meaningful ways" (NCTM 2000, p. 20). The document maintains that new concepts should be introduced, whenever possible, as extensions of familiar mathematics. The activity that follows emphasizes the commonalities among classes of polynomial functions and the themes that are transferred from one class to another.

...Although many teachers see the classes of polynomial functions as extensions of one another, students are prevented from seeing the connections by several instructional factors—the notations used, such as using $P(x)$ only for polynomial functions of degree greater than two; the emphasis on certain attributes of the graphs, for example, the vertex when dealing with parabolas; the emphasis on particular forms of the functions, for example, $y = a(x - h)^2 + k$ for quadratic functions; the techniques used to find the roots, such as factoring quadratic equations; the organization of class activities; and the emphasis on manipulations rather than on understanding. These factors all contribute to forging distinctions rather than commonalities among these functions. Educators need to look beyond entrenched ideas and focus on the connections that students should develop to form a powerful understanding of polynomial functions—connections among the classes of polynomial functions and connections between the graphic and algebraic representations of a function.

... The factor theorem, used in conjunction with the fundamental theorem of algebra, is “glue” that holds the classes of polynomial functions together. These two theorems facilitate the conclusion that any polynomial of degree $n \geq 0$ (with complex coefficients) can be expressed as a product of n linear factors. Therefore, linear factors can be thought of as the building blocks, or components, of all classes of polynomial functions.

The activities that follow draw from this conclusion to make connections among the classes of polynomial functions by graphically focusing on the linear factors of these functions. When the linear factors of each class of polynomial are emphasized,

interpretations of the x-intercept—that is, zeros of a polynomial $f(x)$, roots of the equation $f(x) = 0$, and the intercepts of the graph—become more evident across classes.

A suggestion for assessment is to have students summarize the activity by writing about what they have learned and discussing their understanding of the relationships among linear factors, polynomial functions, and the graphs of these functions. Their comments shed insights into their thinking and help suggest improvements in the activity. A quiz might consist of having the students graph two lines and then sketch the parabola that represents the product of the two linear expressions. Make sure that students explain their reasoning. They could also be instructed to work backward—given the graph of a parabola, they could sketch and find the equations of possible lines that are components of that parabola. I have found that students tend to blur the terms factor and x-intercept. In assessing their writing and oral communication, the teacher should insist that students use the proper terminology and should clarify any "fuzziness" that may exist between these two concepts (Burk, 2000).

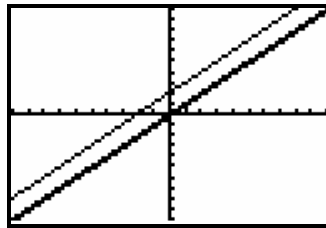
Activity 2: Translations through Families (investigating translations through different families of functions.)

The purpose of this activity is to see the similarities between different families of functions, become familiar with some of the parent graphs of families, and to develop rules for function translation by exploring these things with a graphing calculator. It is also to help the students begin to make predictions of the type of graph and the placement just by looking at the equation so that they can more easily evaluate the reasonableness of their answers. This activity will also help them learn to notice mistakes in inputting equations into the calculator or on homework and to check solutions mentally. This activity also will help make the students comfortable with using their graphing calculator. By having each of the students graph the functions and make their own prediction the idea will be more easily remembered and make more sense.

Materials: Graphing calculators for each of the students. More exact instructions and keystrokes are available in the appendix titled “Translations through Families”

Step 1: Graph the function $y = x$

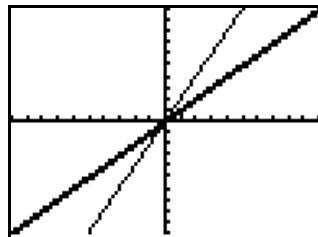
Have the students graph the function $y = x$ (the linear parent function) with a heavy solid line; at the same time have the students graph the function $y = x + 2$ and leave the line normal. This enables the students to see which function is which. It may be helpful to have the students sketch the graphs on their notes as they go through so they can keep track of what they are learning. The graph should look like:



Ask the students to explain what happened to the graph. (It shifts two up or it shifts two to the left)

Step 2: Graph $y = 2x$

Then have the students clear the second graph ($y = x + 2$) and graph the function $y = 2x$ in its place; the graph should look like:



Ask the students to explain what happened. (It gets steeper)

Step 3: Graph other types of translations from rules.

Then try $y = \frac{1}{2}x$, $y = x - 3$, $y = -x$, $y = 2x + 1$, $y = -\frac{1}{2}x + 3$, each separately with the parent function. See if they can come up with general rules; let them put it in their own words but guide them to these conclusions: “The number added after shifts the graph up and down depending on the sign” “the number in front of the x makes the graph steeper or flatter, bigger = steeper, smaller = flatter;” and, “The positive or the negative in front of the x determines whether the graph rises or falls, positive = rises, negative = falls.”

Step 4: Make predictions, then graph

Give them an equation like $y = \frac{1}{2}x - 4$ and ask them to predict what will happen to the parent graph.

Step 5: Repeat with different families of functions

Repeat the steps above for $y = x^2$ as the parent graph. On the graph have the students graph the functions $y = x^2 + 4$, $y = (x + 3)^2$, $y = 2x^2$, $y = (x - 5)^2$, $y = \frac{2}{3}x^2$, $y = x^2 - 4$ separately and notice the changes in each. Have them write rules something like this:

“inside \rightarrow opposite direction left and right;” “outside \rightarrow same direction up and down;”
 “positive $x^2 =$ happy, can hold water, curving up;” “negative $x^2 =$ sad, can not hold water, curving down;” “big number in front of $x^2 =$ steeper, thinner, stretched up;” and, “smaller number in front of $x^2 =$ flatter, fatter, stretched out”

Then follow the same procedure using the family of absolute value, the family of larger polynomials, the square root family, the log family, and the exponential family. They should be able to generalize the rules to apply to any family of functions.

Activity 3: Mix and match functions (Be able to determine just from looking at an equation what the graph will look like)

This activity should reinforce the idea of translations and help the students be able to determine the different types of functions and how to predict what a function will look like just by looking at the equation.

Materials: A set of graphs with equations for those graphs separately, and graphing calculator to check solutions. (A possible set is available in the appendix titled “Mix and Match Functions”).

Step 1: Distribute the materials

Give the students the “Mix and Match Functions” sheet and in groups have them complete it without the help of the calculator. Then tell students to check their solutions with the calculator.

VII. CONCLUSION

These activities were designed to follow the recommendations of well-respected learning theories and theorists. Considering students' intellectual development influenced the order of activities presented, the vocabulary used in the activities, and focused on the idea of helping students construct their own understanding. Learning styles, modalities, intelligences, and attention to environment, verify the diverse types of activities employed.

These activities have not been field-tested. I am not qualified to do any quantitative testing to determine the effectiveness of these materials, nor am I qualified to analyze test scores to find these struggle areas or verify the usefulness of any of the materials in this work. However, following the advice of educational theorists increases chance for success in the employment of this thesis.

Until I have a class of my own it is unlikely that any of these materials will be used. However, this will soon happen, I am looking forward to seeing results, and I feel the process was beneficial in that by taking such a close look at so important a topic in mathematics that I will be able to better help my students understand.

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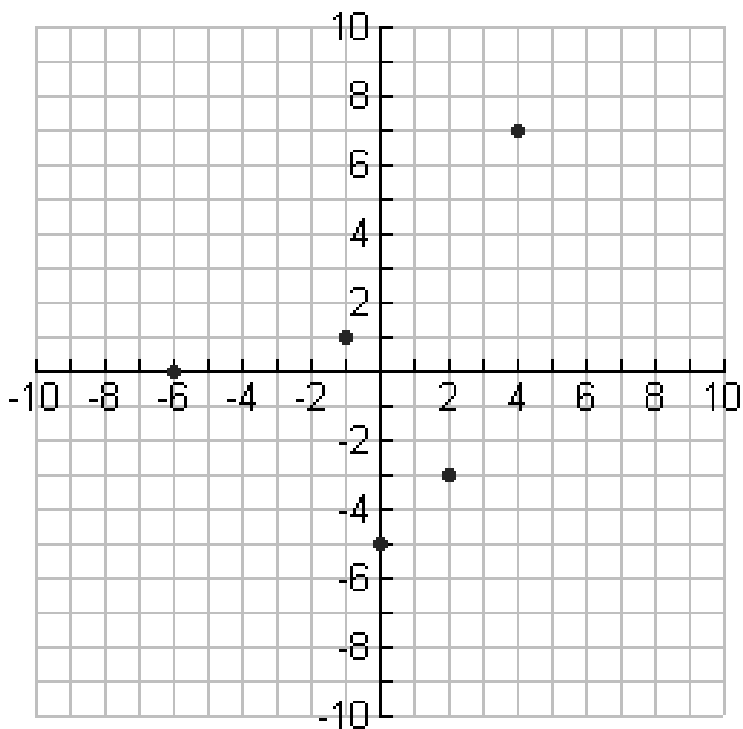
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Appendix

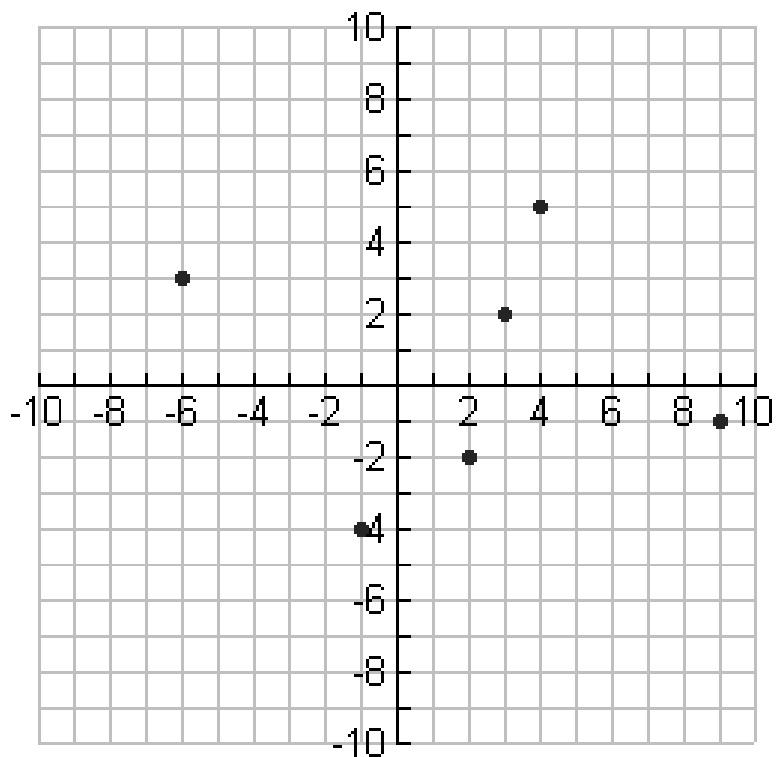
“Find It” Sets of graphs	Domain and Range Activity 2	68-80
“Restricted Domain?” Calculator instructions	Domain and Range Activity 3	81-82
“Fun Plugging into Functions” Function Cards	Function Manipulation Activity 1	83-88
“If given the domain, Find the range” Sheets for finding range graphically	Function Manipulation Activity 5	89-91
“Function Composition” Step-by-Step guidance sheet	Function Composition Activity 1	92
“Function Inverse” Step-by-Step guidance sheet	Function Composition Activity 3	93
“Functions and Inverses” Graphs for Mira Activity	Function Composition Activity 4	94-97
“Building Polynomial Functions” Sheets and answers by Judy Burk	Families of Functions Activity 1	98-103
“Translations through Families” Calculator instructions	Families of Functions Activity 2	104-105
“Mix and Match Functions” Sheets to practice translations	Families of Functions Activity 3	106-108

Find It!

$$\{(-6,0), (-1, 1), (0, -5), (2, -3), (4,7)\}$$

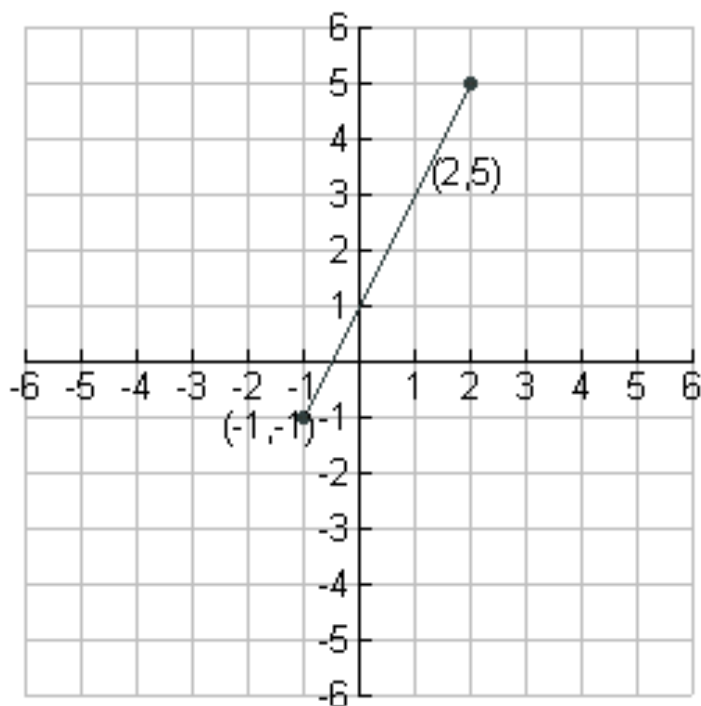


$$\{(3,2), (2,-2), (4,5), (-1,-4), (-6,3), (9,-1)\}$$

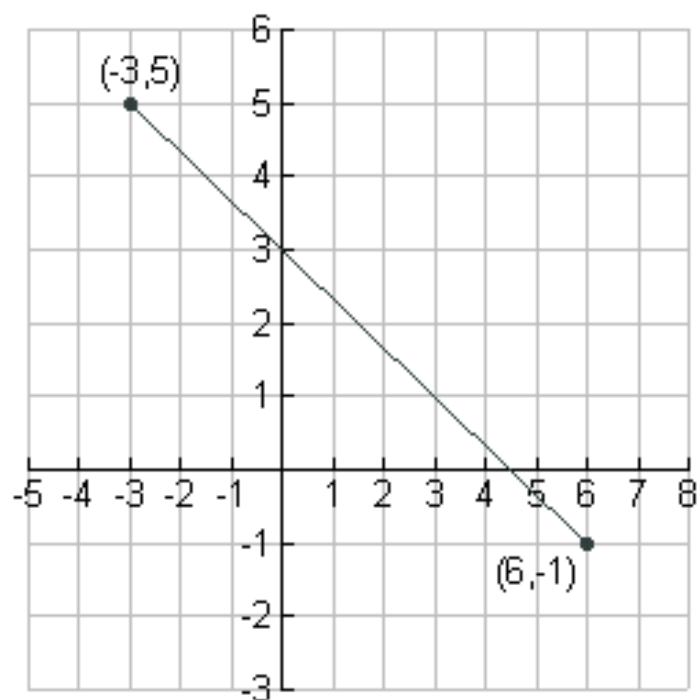


Find It!

$$y = 2x + 1$$

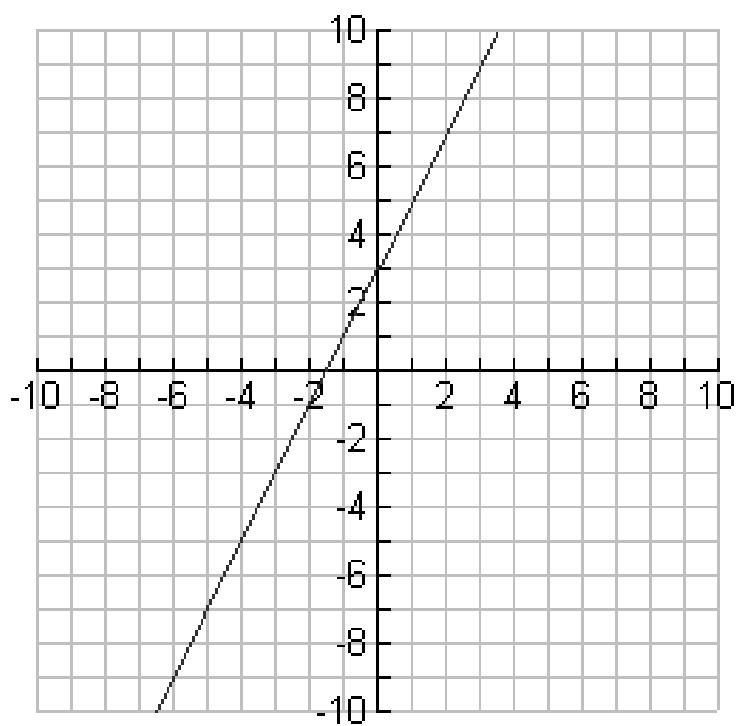


$$y = \frac{-2}{3}x + 3$$

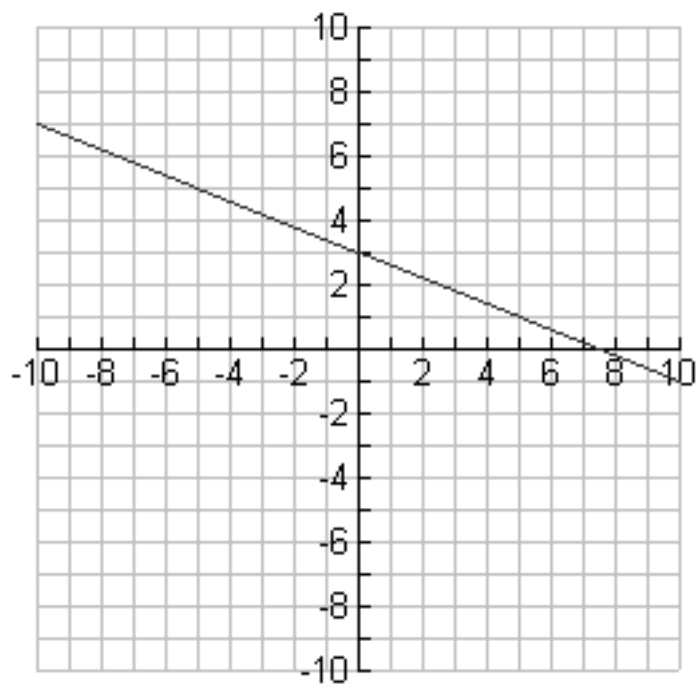


Find It!

$$y = 2x + 3$$

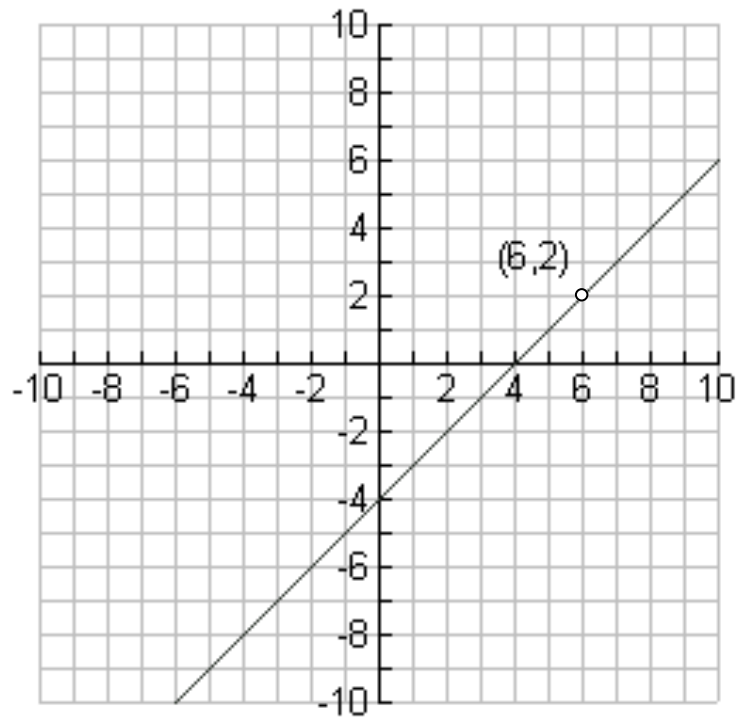


$$y = \frac{-2}{5}x + 3$$

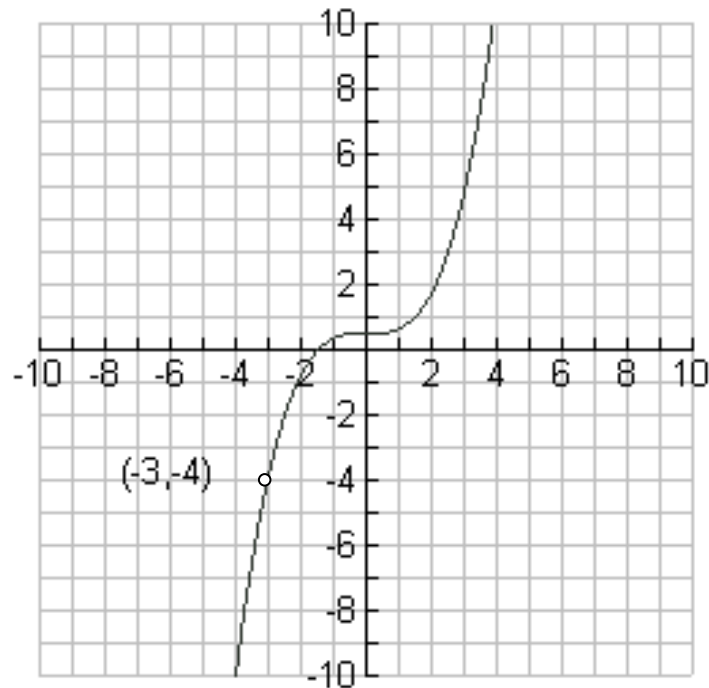


Find It!

$$y = x - 4 \quad \text{With hole } (6,2)$$

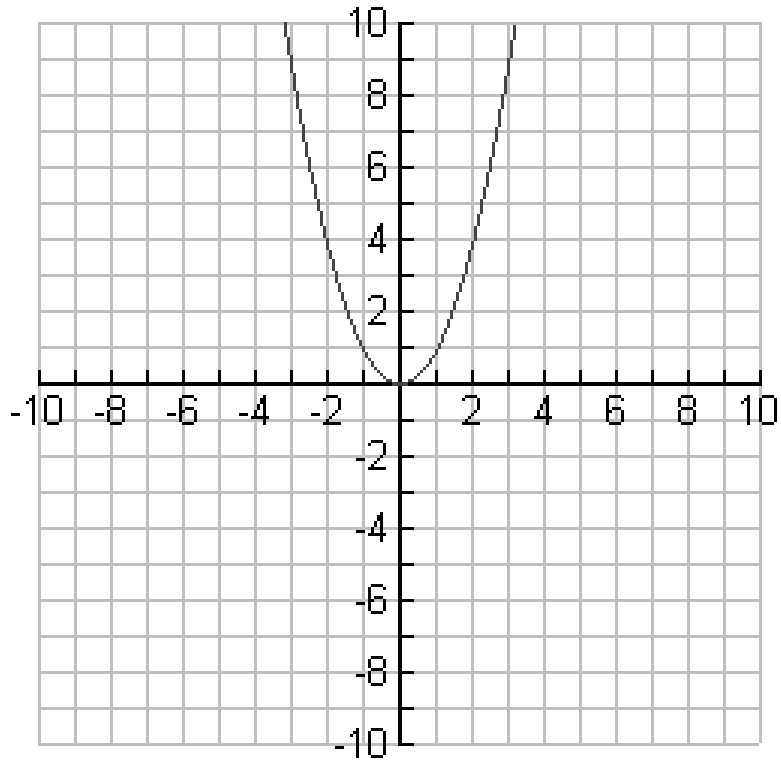


$$y = \frac{1}{6}x^3 + \frac{1}{2} \quad \text{With hole } (-3, -4)$$

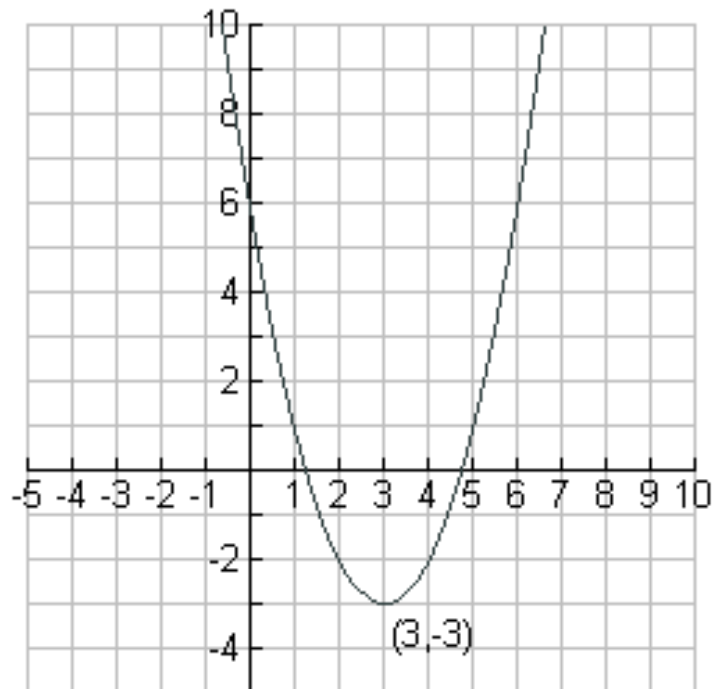


Find It!

$$y = x^2$$

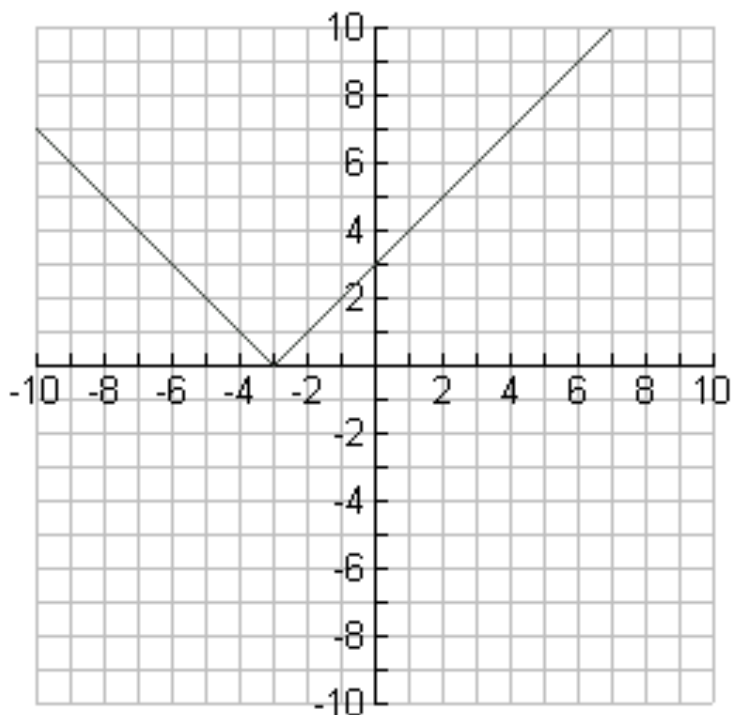


$$y = (x - 3)^2 - 3$$

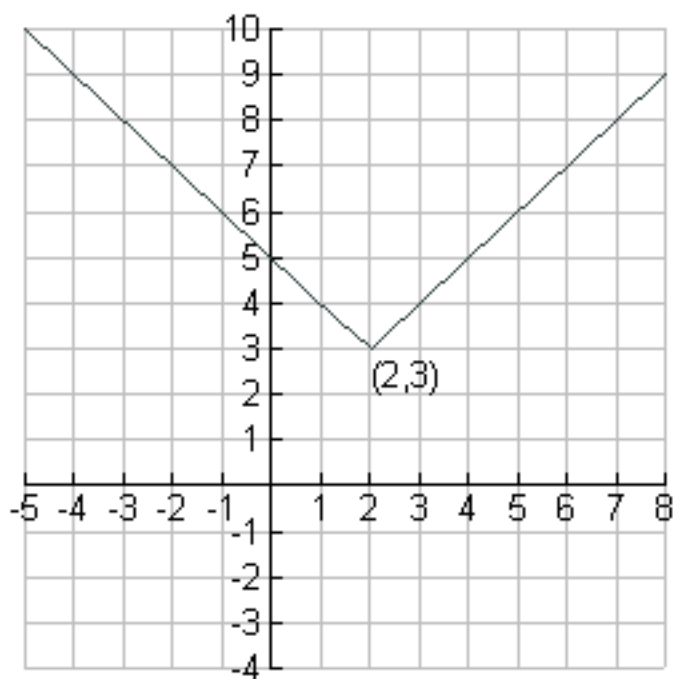


Find It!

$$y = |x + 3|$$

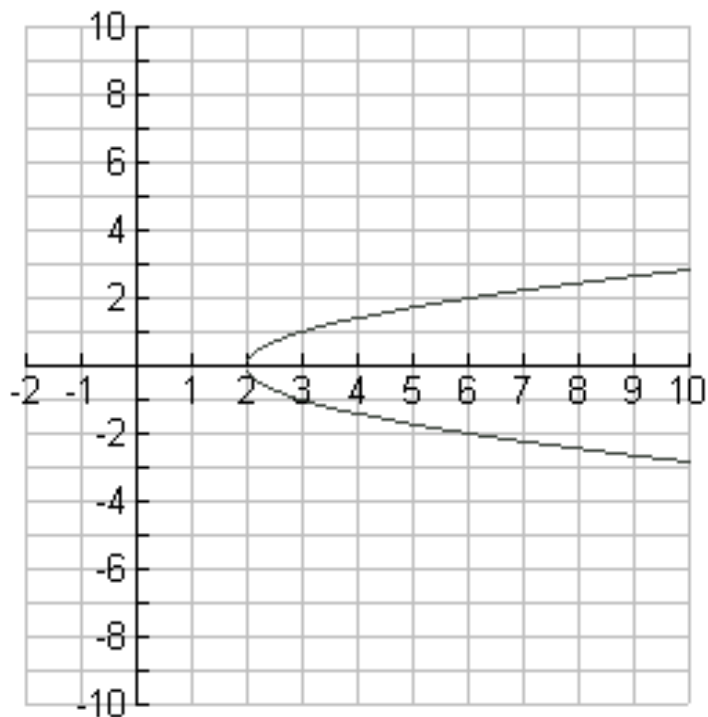


$$y = |x - 2| + 3$$

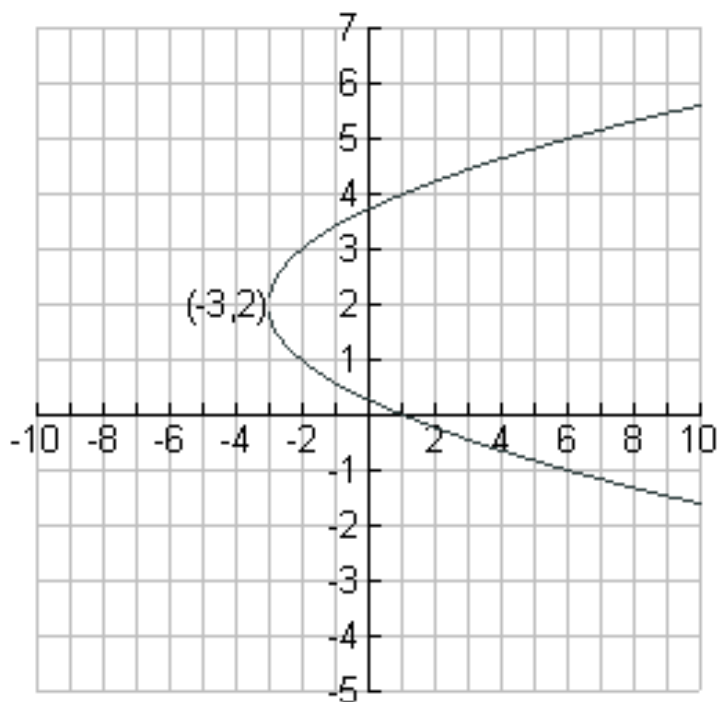


Find It!

$$x = y^2 + 2$$

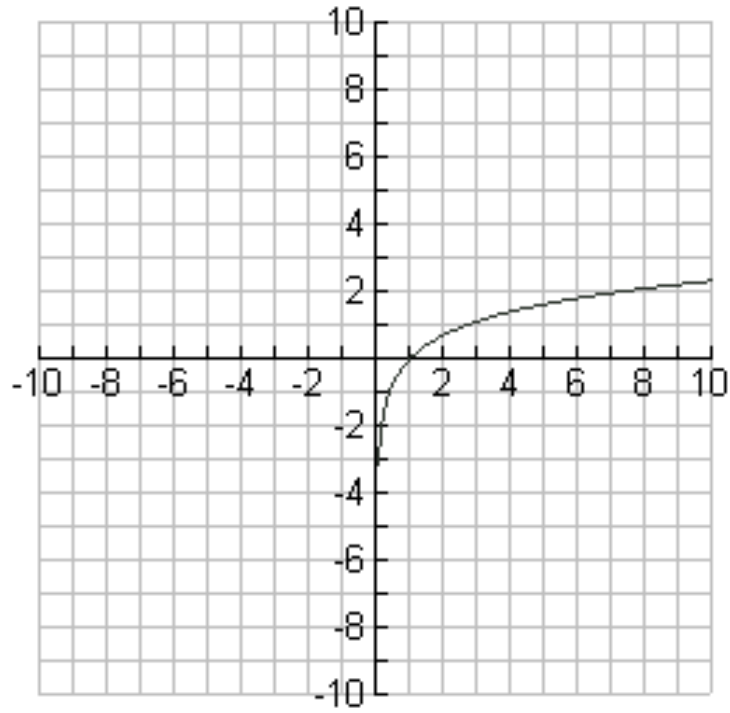


$$x = (y - 2)^2 - 3$$

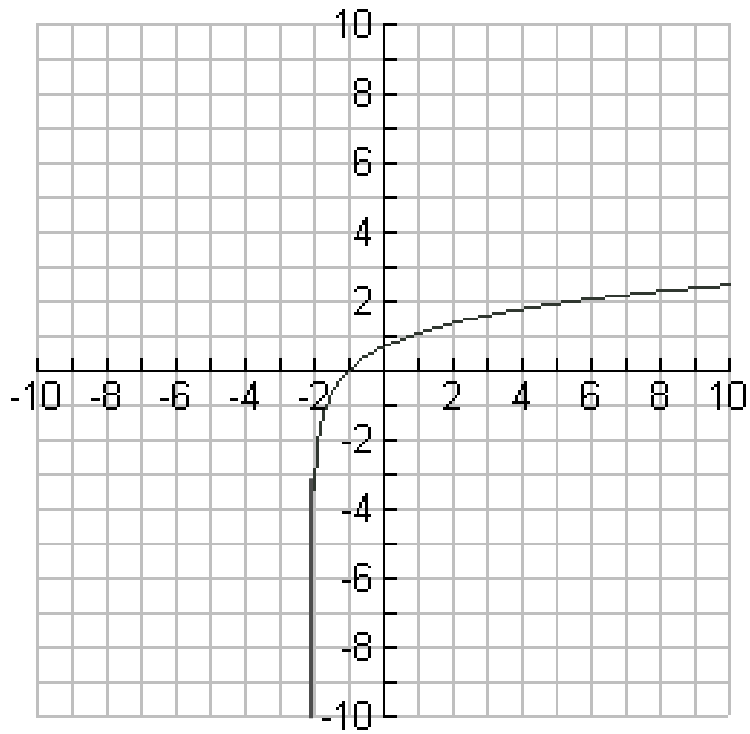


Find It!

$$y = \ln(x)$$

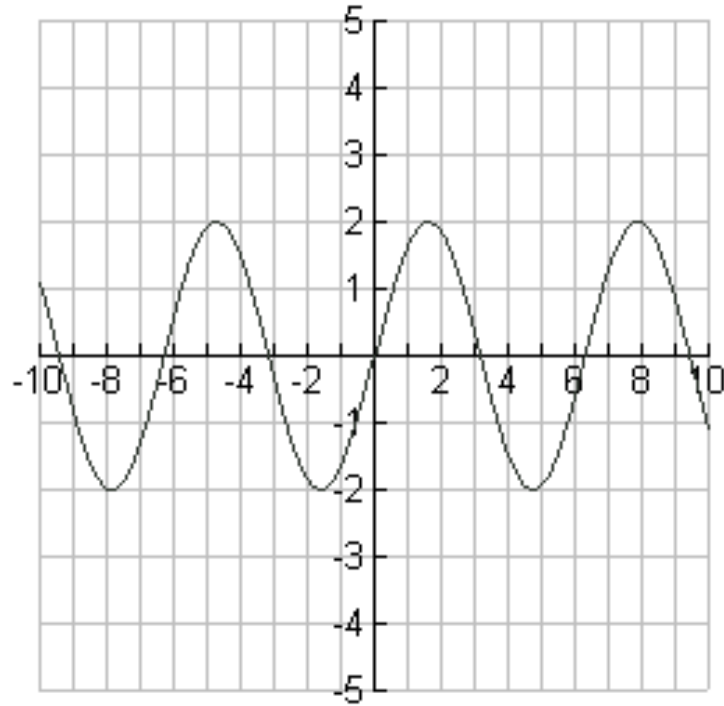


$$y = \ln(x + 2)$$

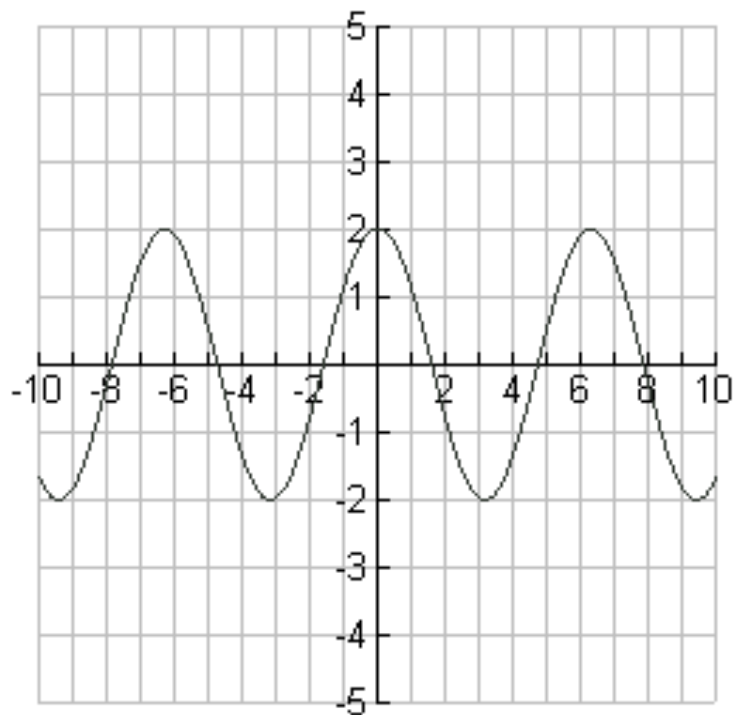


Find It!

$$y = 2 \sin(x)$$

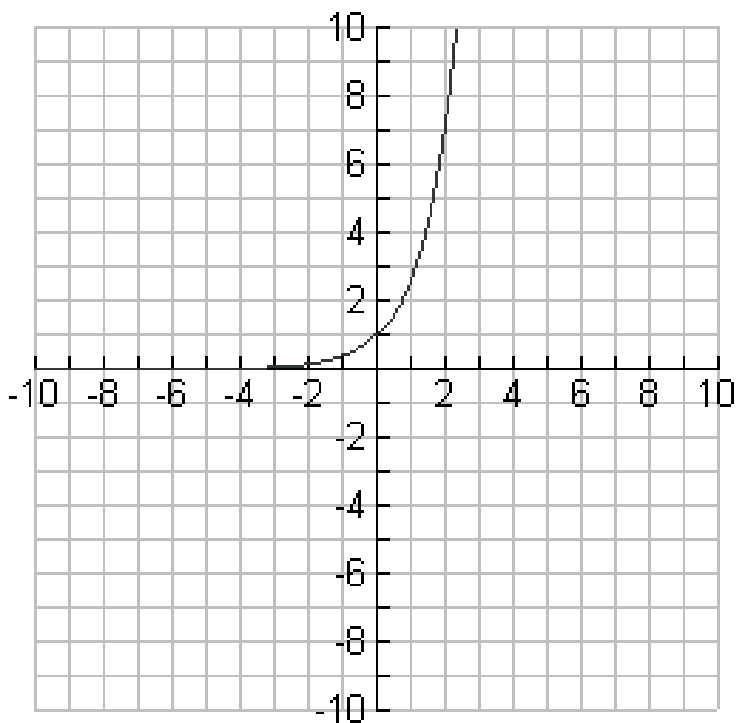


$$y = 2 \cos(x)$$

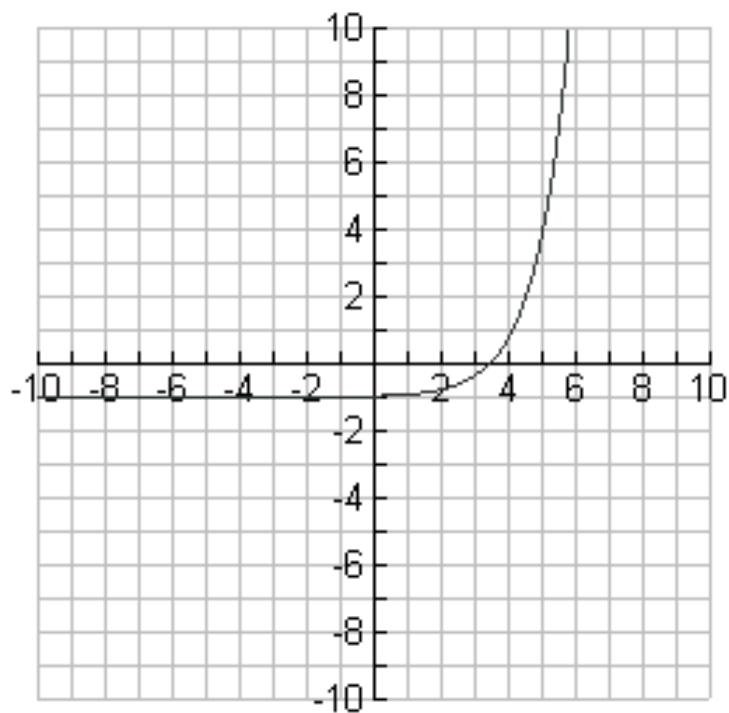


Find It!

$$y = e^x$$

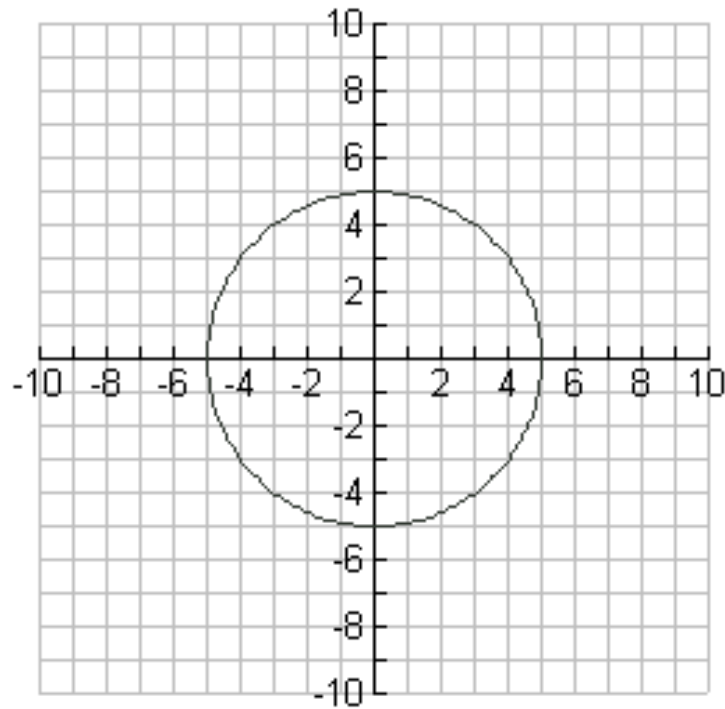


$$y = \frac{1}{4}e^{x-2} - 1$$

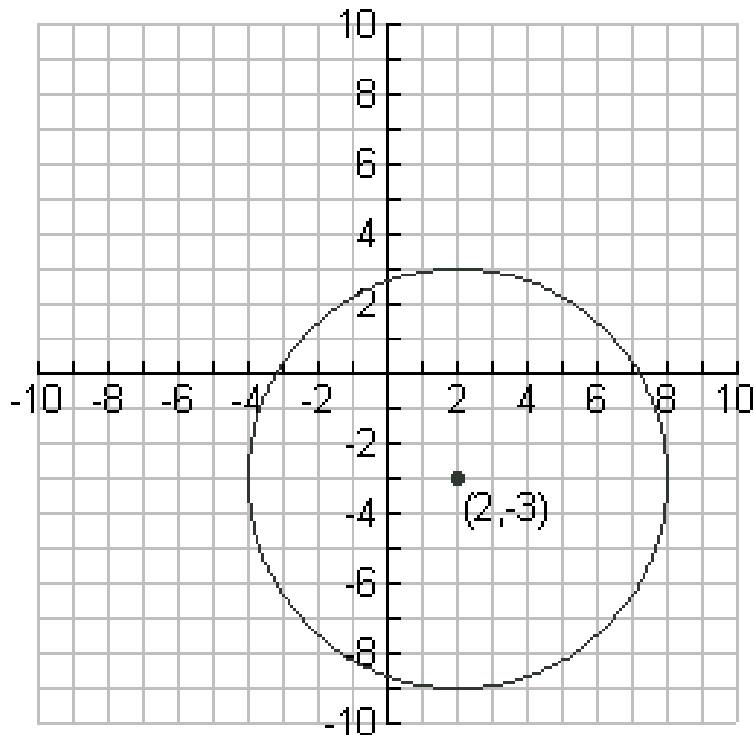


Find It!

$$x^2 + y^2 = 25$$

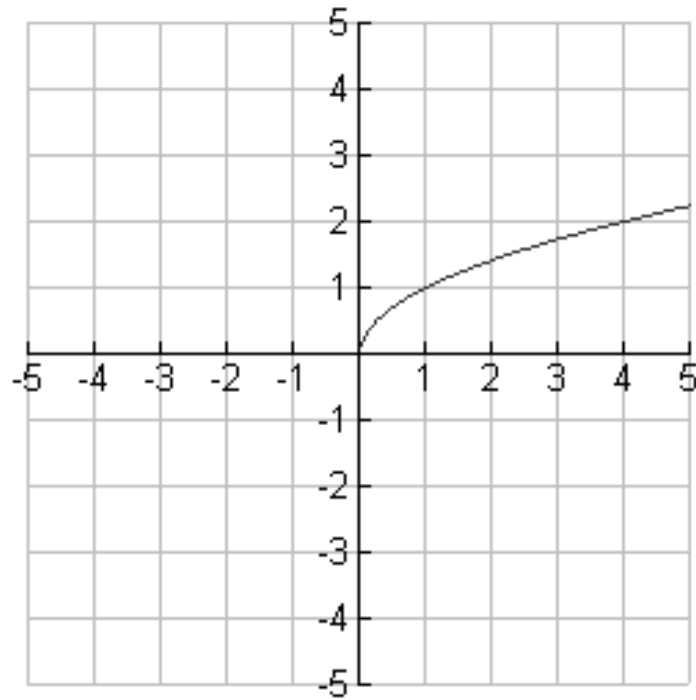


$$(y + 3)^2 + (x - 2)^2 = 36$$

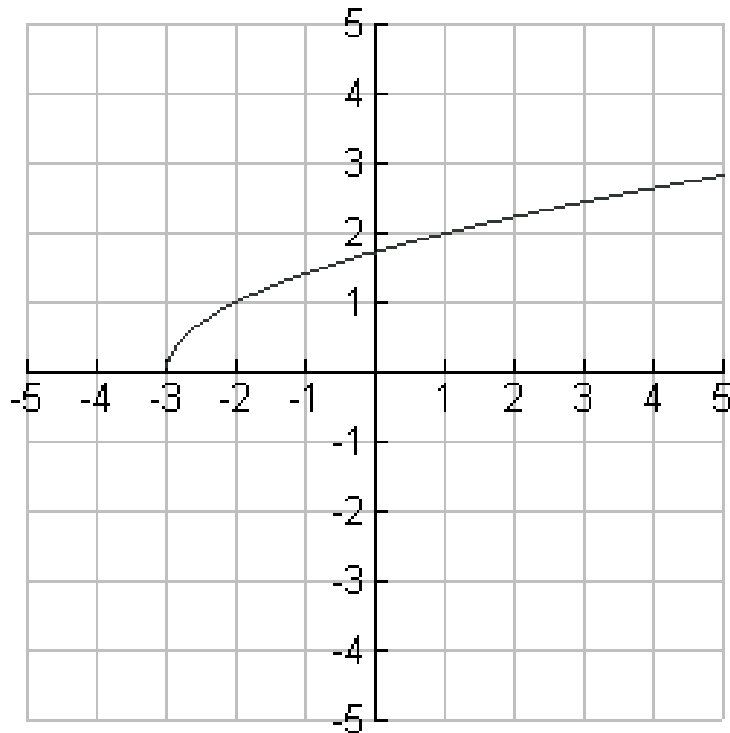


Find It!

$$y = \sqrt{x}$$



$$y = \sqrt{x+3}$$



Restricted Domain?

Calculator Instructions

First enter the equation into the graphing screen by pushing \square then \square . Your screen should look like:

```

Plot1 Plot2 Plot3
\Y1=
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

```

Into Y_1 enter an equation, for example the equation $y = \sqrt{x+3}$.

Move the cursor until the Y_1 is highlighted then push \square to access the square root ($\sqrt{\quad}$) function, which shows yellow above \square . Then push \square to put in the variable “X” \square to finish entering the function. Your screen should look like:

```

Plot1 Plot2 Plot3
\Y1=√(X-3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

```

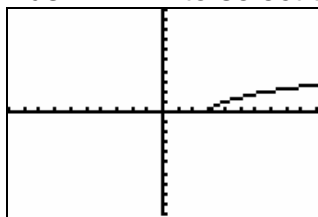
Push \square to set the window and using the arrow keys move the cursor down until you reach option 6: which says “ZSTANDARD”

```

ZOOM MEMORY
1:ZBox
2:Zoom In
3:Zoom Out
4:ZDecimal
5:ZSquare
6:ZStandard
7↓ZTrig

```

Push \square to select then push \square to see the graph on a 10X10 screen.



The students should be able to see from this that the domain does not exist less than 3. Have them check their predictions using the function ability explained below.

Push \square to access “Quit” which appears above \square in yellow to clear the screen.

Then push \square , this stands for variables and gives you direct access to the functions. Use the arrow keys to move the cursor over until Y-VARS is highlighted. The screen should look like:

```

VARS VARS
1:Function...
2:Parametric...
3:Polar...
4:On/Off...

```

Push **ENTER** to select the option “Function”.

```

FUNCTION
1:Y1
2:Y2
3:Y3
4:Y4
5:Y5
6:Y6
7↓Y7

```

Push **ENTER** to select the option “Y₁” because that is where we entered the function earlier.

```

Y1 █

```

To attempt a domain element for example “3” Push **(3)** **ENTER**

We know that three is in the domain and we expect to get 0, which we do.

```

Y1(3)
0

```

To try another domain element, for example “2” which we know is not in the domain of the function go back to the **VAR**S step to get the function again, this time push **(2)**.

```

Y1(3)
0
Y1(2) █

```

When you push **ENTER** the screen should look like:

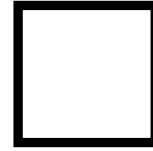
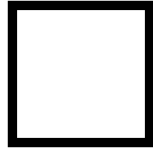
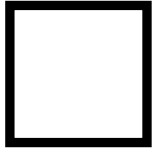
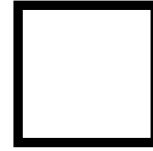
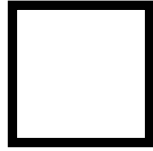
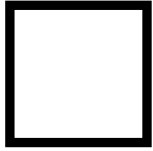
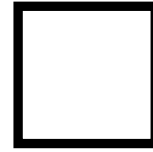
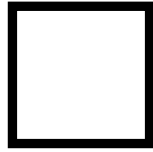
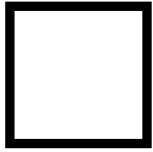
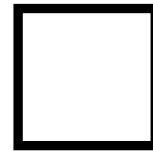
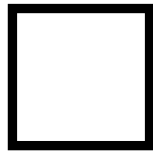
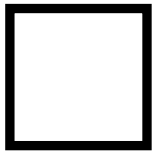
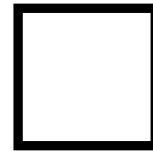
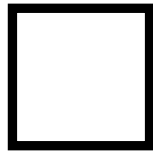
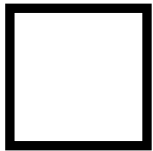
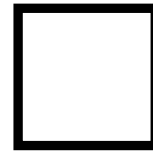
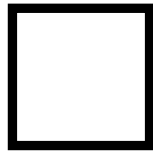
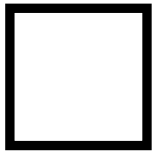
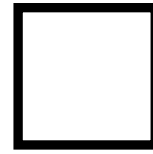
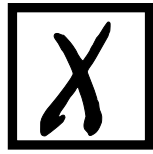
```

ERR:NONREAL ANS
1:Quit
2:Goto

```

This shows that 2 is not in the domain of the function.

Fun Plugging into Functions



Function _____

$$F(\square) = 2\square + 4 = \square$$

1

$$G(\square) = \frac{2}{3}\square - 2 = \square$$

2

$$H(\square) = -5\square + 3 = \square$$

3

$$E(\square) = 9\square + 1 = \square$$

4

$$Y(\square) = -3\square + \frac{2}{5} = \square$$

5

$$Z(\square) = \frac{-3}{10}\square - \frac{1}{2} = \square$$

6

$$W(\square) = 7\square - 14 = \square$$

7

$$V(\square) = 4\square + 5 = \square$$

8

$$A(\square) = -5\square + \frac{1}{4} = \square$$

9

$$B(\square) = \frac{3}{10}\square - 5 = \square$$

10

$$C(\square) = -3\square + 2 = \square$$

11

$$D(\square) = \frac{-4}{9}\square + \frac{1}{3} = \square$$

12

$$Q(\square) = -7\square + \frac{5}{6} = \square$$

13

$$R(\square) = \frac{7}{10}\square + \frac{3}{5} = \square$$

14

$$S(\square) = -2\square + 5 = \square$$

15

$$T(\square) = \frac{-5}{7}\square + 6 = \square$$

16

$$L(\square) = -3\square + \frac{5}{12} = \square$$

17

$$M(\square) = \frac{1}{4}\square + \frac{5}{6} = \square$$

18

$$N(\square) = -3\square + 7 = \square$$

19

$$O(\square) = -\frac{1}{3}\square + 2 = \square$$

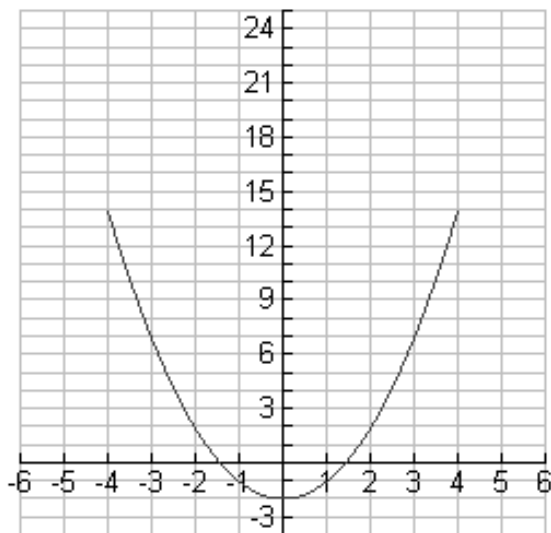
20

Name _____

If Given the Domain, find the Range

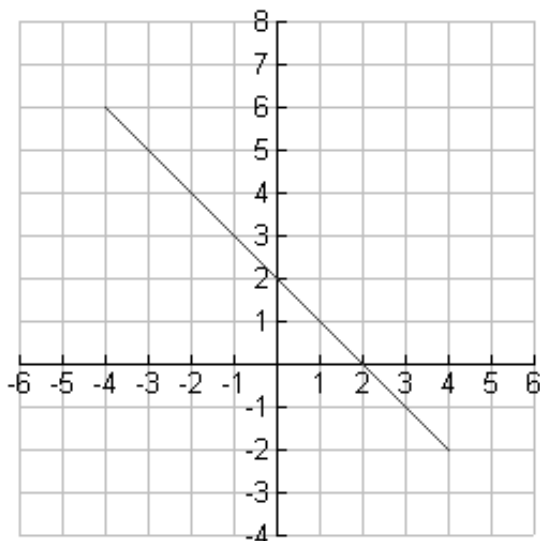
Directions: Using the corresponding functions and matching graphs to answer the questions that follow, you may use your calculator if necessary.

1. $f(x) = x^2 - 2$



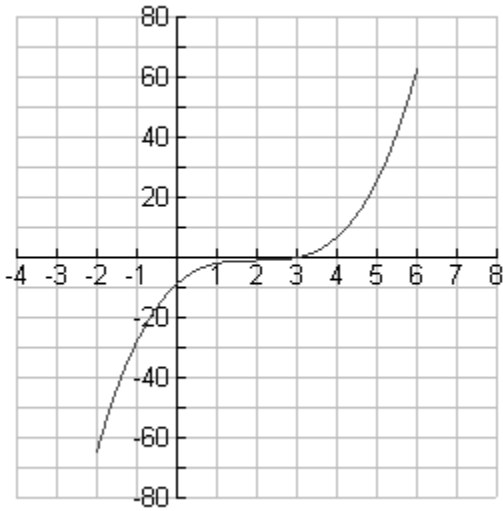
- a) Find $f(-3)$. a) _____
- b) Find $f(3)$. b) _____
- c) What do you notice about a) and b)? c) _____
- d) Find $f(0)$. d) _____
- e) How does d) relate to the function? e) _____
- f) What values of x correspond to 14? f) _____
- g) Find $f(-5)$ algebraically. g) _____
- Check yourself by continuing the graph.
- h) Using what you know from a) and b), What is $f(5)$? h) _____

2. $g(x) = -x + 2$



- a) Find $g(-1)$. a) _____
- b) Find $g(2)$. b) _____
- c) Find $g(0)$. c) _____
- d) How does b) relate to the function? d) _____
- d) How does c) relate to the function? e) _____
- e) What value of x corresponds to -2? f) _____
- f) Find $g(5)$ algebraically. g) _____
- Check yourself by continuing the graph.

3. $h(x) = (x-2)^3 - 1$



- | | |
|--|----------|
| a) Approximate $h(2)$ using the graph. | a) _____ |
| b) Find $h(2)$ algebraically. | b) _____ |
| c) How does b) relate to the function? | c) _____ |
| d) Find $h(0)$ using the graph. | d) _____ |
| e) Find $h(0)$ algebraically. | e) _____ |
| f) What value of x corresponds to 0 using the graph? | f) _____ |
| g) Does this value give an answer of 0 when you check it algebraically? If not find the value that does? | g) _____ |
| h) Find $f(-5)$ algebraically. | h) _____ |
| i) Is it logical to continue the graph with this value? | i) _____ |

Is it practical to find the range value from the domain value by simply using the graph in this case? Why or why not?

Is the graph useful at all in this case? If it is, what are the benefits? If it is not what would be useful?

Answers:

1. $f(x) = x^2 - 2$

- | | |
|---|--|
| a) Find $f(-3)$. | a) 7 |
| b) Find $f(3)$. | b) 7 |
| c) What do you notice about a) and b)? | c) They are the same |
| d) Find $f(0)$. | d) -2 |
| e) How does d) relate to the function? | e) It is the constant term (or similar answer) |
| f) What values of x correspond to 14? | f) -4 and 4 |
| g) Find $f(-5)$ algebraically. | g) 23 |
- Check yourself by continuing the graph. (The students can do this right on the paper)
- | | |
|--|-------|
| h) Using what you know from a) and b),
What is $f(5)$? | h) 23 |
|--|-------|

2. $g(x) = -x + 2$

- | | |
|---|--|
| a) Find $g(-1)$. | a) 3 |
| b) Find $g(2)$. | b) 0 |
| c) Find $g(0)$. | c) 2 |
| d) How does c) relate to the function? | d) It is the constant term (or similar answer) |
| e) How does b) relate to the function? | e) It is where the graph crosses the x-axis |
| f) What value of x corresponds to -2? | f) 4 |
| g) Find $g(5)$ algebraically. | g) -3 |
- Check yourself by continuing the graph. (the students can do this right on the paper)

3. $h(x) = (x-2)^3 - 1$

- | | |
|--|--|
| a) Approximate $h(2)$ using the graph. | a) Anywhere between 0 and -4 |
| b) Find $h(2)$ algebraically. | b) -1 |
| c) How does b) relate to the function? | c) It is after the $(x-2)^3$ (or similar answer) |
| d) Find $h(0)$ using the graph. | d) Anywhere between -6 and -10 |
| e) Find $h(0)$ algebraically. | e) -9 |
| f) What value of x corresponds
to 0 using the graph? | f) Anywhere between 1-3 (however they
should know from b) that it is not 2. |
| g) Does this value give an answer
of 0 when you check it algebraically?
If not find the value that does. | g) Check f) if $f) = 3$ yes; if $f) \neq 3$ no
3 |
| h) Find $f(-5)$ algebraically. | h) -344 |
| i) Is it logical to continue the graph
with this value? | i) no |

Is it practical to find the range value from the domain with this type of graph? Why or why not?

Answers vary. Should be mostly no because the scale on the y-axis has to get too large too quickly to accurately find values.

Is the graph useful at all in this case? If it is, what are the benefits? If it is not what would be useful?

Answers vary. If yes, the graph is good for checking answers or making approximations. If no, see what they answer you may be able to use their answers to make more useful tools later.

Function Composition

Name _____

$$f(x) = 2x - 3 \text{ and } g(x) = 4x - 1$$

What to Find.	Step 1: rewrite and simplify	Step 2: replace	Step 3: plug in	Step 4: simplify and solve	Step 5: find if $x=2$
fog(x)	$fog(x) = f(g(x))$ $= f(g)$	$f(g) = 2g - 3$	$f(g(x)) = 2(4x - 1) - 3$	$f(g(x)) = 8x - 2 - 3$ $fog(x) = 8x - 5$	$fog(2) = 8(2) - 5$ $fog(2) = 11$
fof(x)					
gof(x)					
gog(x)					

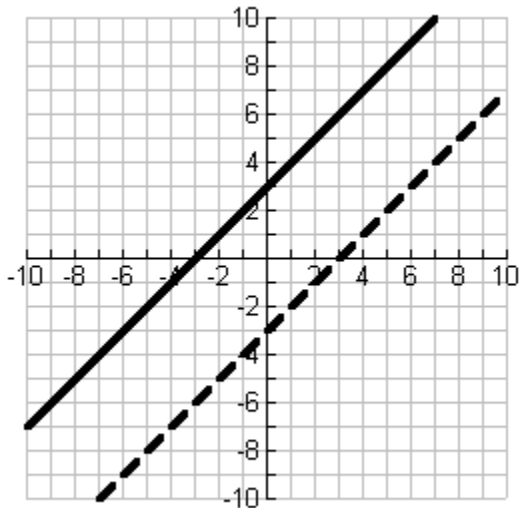
Finding the Function Inverse

Name _____

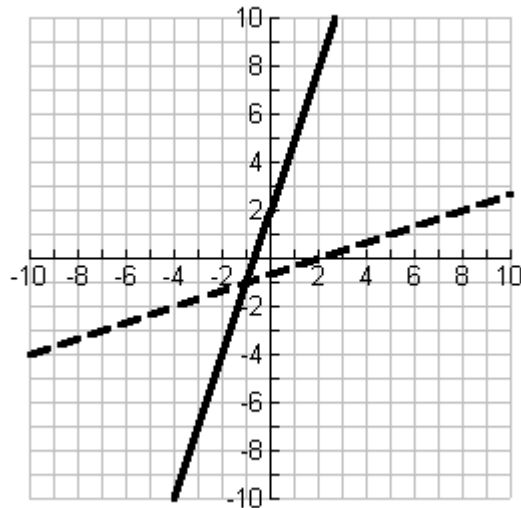
Function:	$f(x) = x + 3$				
Step 1: Simplify name	$f = x + 3$				
Step 2: Switch variables	$x = f + 3$ _____				
Step 3: solve for new domain	$x = f + 3$ -3 -3 $x - 3 = f$ $f = x - 3$				
Step 4: Rename	$f^{-1}(x) = x - 3$				

Graphs of Functions and their Inverses

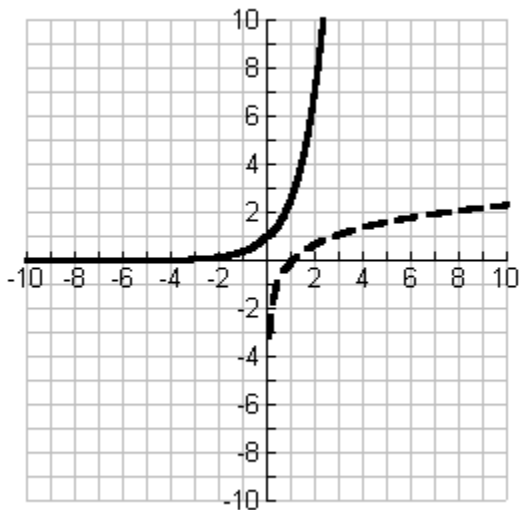
$$f(x) = x + 3 \text{ and } f^{-1}(x) = x - 3$$



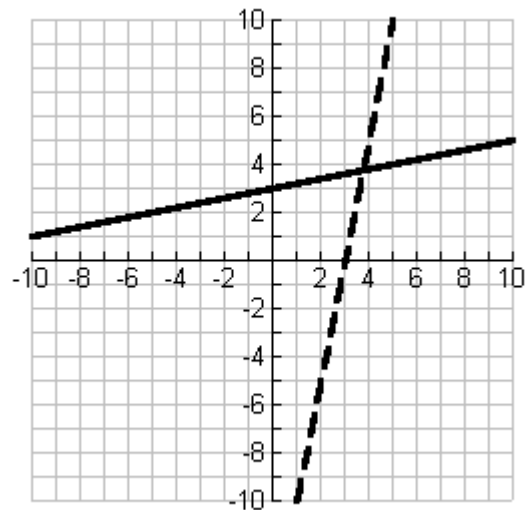
$$f(x) = 3x + 2 \text{ and } f^{-1}(x) = \frac{x - 2}{3}$$



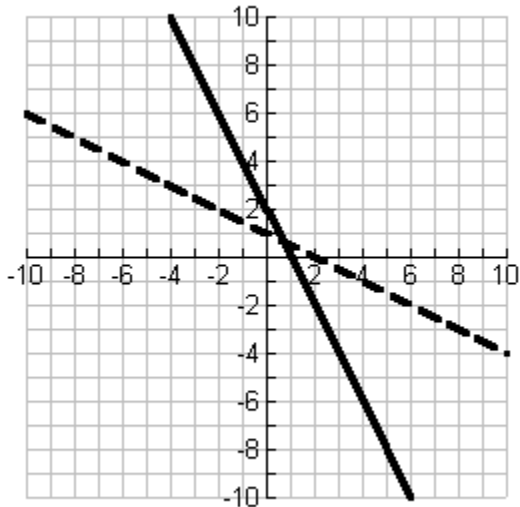
$$f(x) = e^x \text{ and } f^{-1}(x) = \ln(x)$$



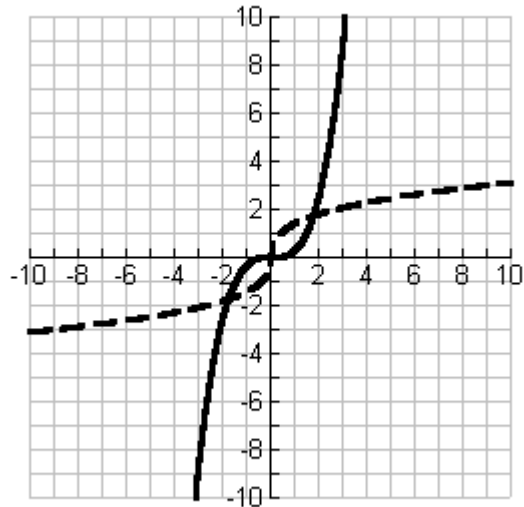
$$f(x) = \frac{x}{5} + 3 \text{ and } f^{-1}(x) = 5(x - 3)$$



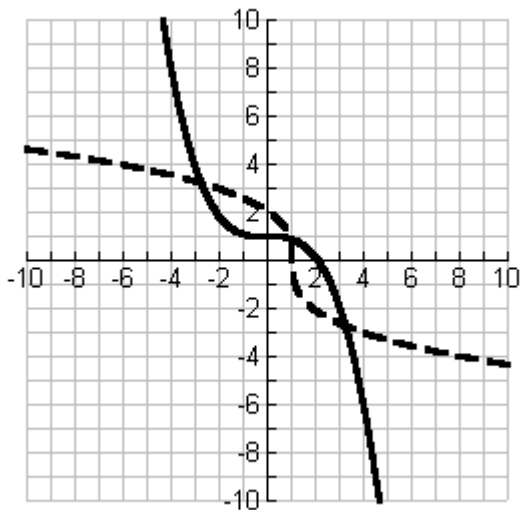
$$f(x) = -2x + 2 \text{ and } f^{-1}(x) = -\frac{x-2}{2}$$



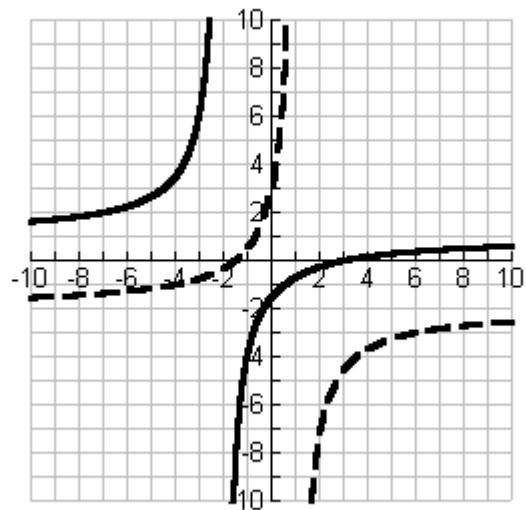
$$f(x) = \frac{x^3}{3} \text{ and } f^{-1}(x) = (3x)^{\frac{1}{3}}$$



$$f(x) = \frac{-x^3}{9} + 1 \text{ and } f^{-1}(x) = (9 - 9x)^{\frac{1}{3}}$$

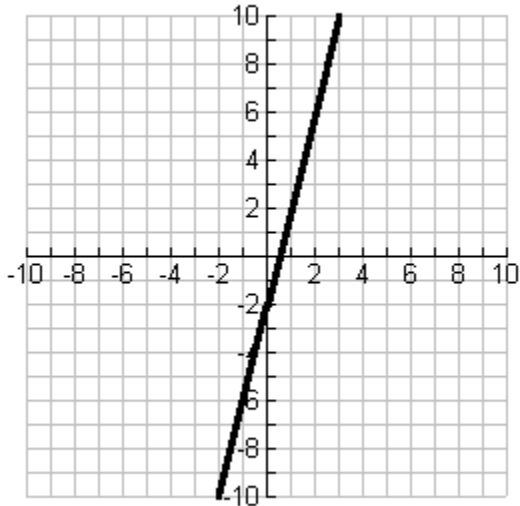


$$f(x) = \frac{x-3}{x+2} \text{ and } f^{-1}(x) = -\frac{2x+3}{x-1}$$

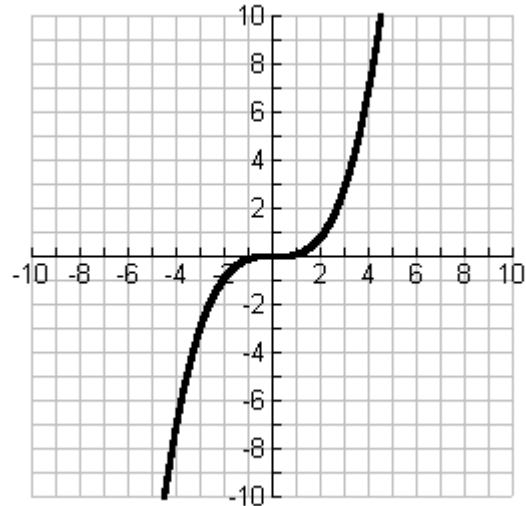


Draw the Inverse, Is the inverse a Function?

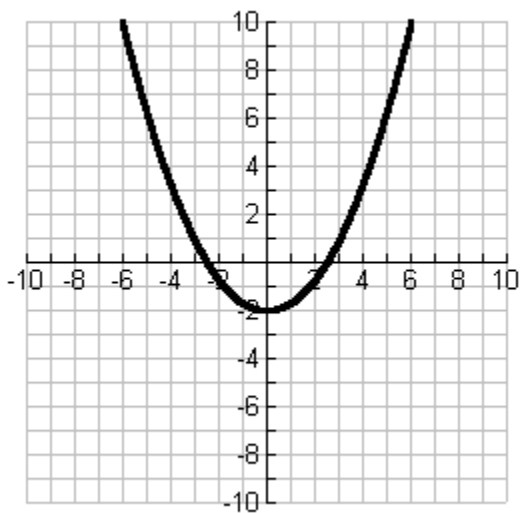
$$f(x) = 4x - 2$$



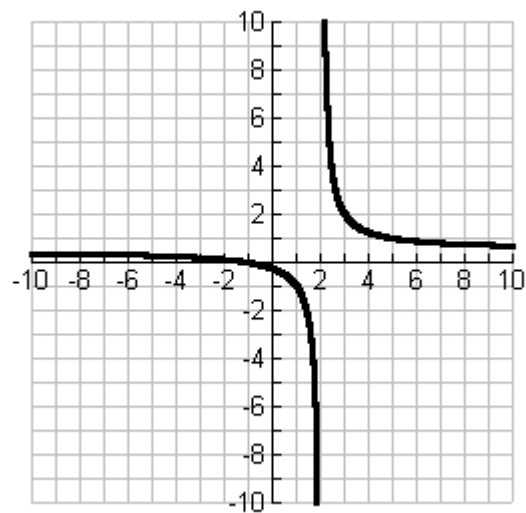
$$f(x) = \frac{x^3}{9}$$



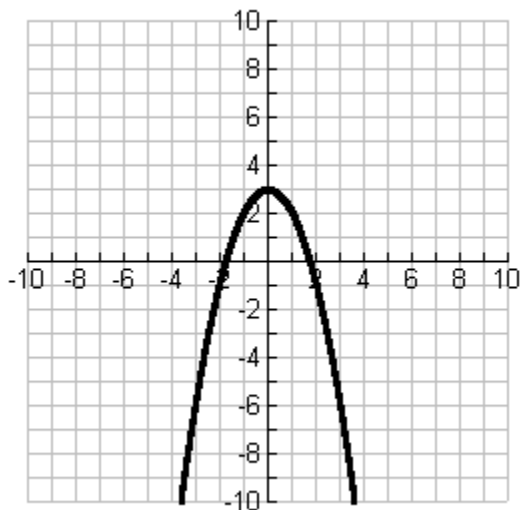
$$f(x) = \frac{x^2}{3} - 2$$



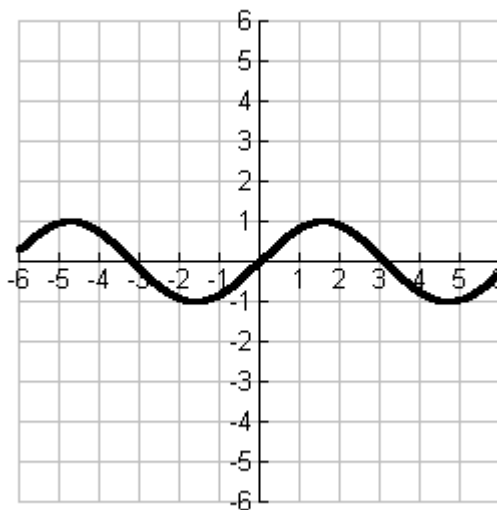
$$f(x) = \frac{x+1}{2x-4}$$



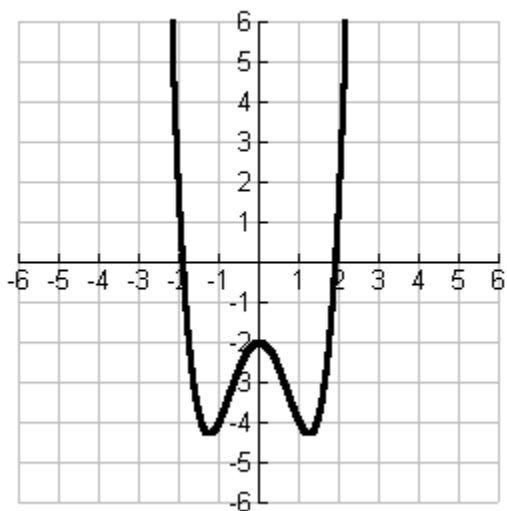
$$f(x) = -x^2 + 3$$



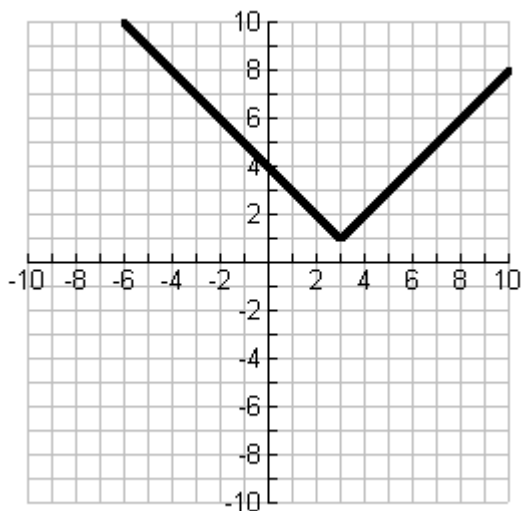
$$f(x) = \sin(x)$$



$$f(x) = x^4 - 3x^2 - 2$$



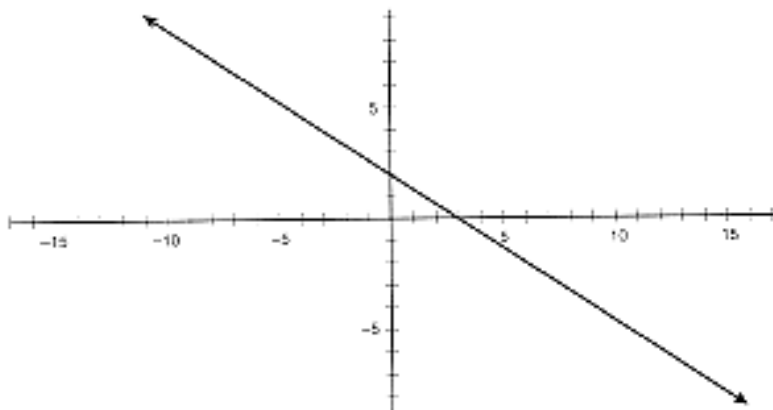
$$f(x) = |x - 3| + 1$$



BUILDING POLYNOMIAL FUNCTIONS

SHEET 1A

1. What is the equation of the following linear function? _____



2. How did you find it?
3. The slope/ y -intercept form of a linear function, $y = mx + b$, is commonly used to find the equation of the function. If your function is not in this form, put it in this form.
4. Rewrite the linear function by factoring out the slope, m , that is, let $y = m(x + b/m)$.
5. Choose a different linear function written in the form $y = mx + b$, where $m \neq 0$, and graph it on the axes shown for question 1. Use a different color to sketch this line. Rewrite this equation in the form $y = m(x + b/m)$.
6. For each function, what does $-b/m$ represent, that is, what is the connection between these values and the graph of each linear function?

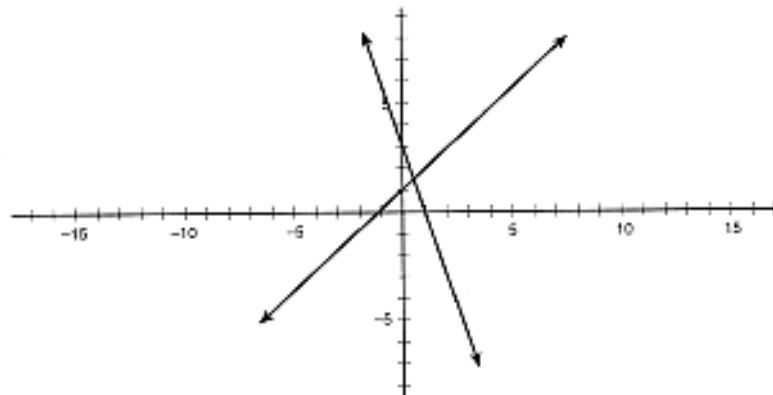
Letting $c = -b/m$, the form $y = m(x - c)$ might be called the *slope/ x -intercept* form of the equation, where c is the x -intercept of the line. The factor theorem states that if c is a root (x -intercept) of a polynomial function, then $(x - c)$ must be a factor of that polynomial function. Note that $(x - c)$ is a factor of the expression. The only other factor is the slope, m .

7. On the axes given in question 1, sketch the graph of the function obtained by taking the product of the two linear expressions.
8. Did you sketch a particular type of graph? Describe it.
9. Find the product of the two expressions, and graph the corresponding quadratic function. Does it match the parabola that you sketched?
10. What do you notice about the parabola in relation to the lines that were its components?

BUILDING POLYNOMIAL FUNCTIONS—*Continued*

SHEET 1B

11. Describe the relationship between the x -intercepts of the lines and the x -intercept(s) of the parabola.
12. Describe the relationship between the y -intercepts of the lines and the y -intercept(s) of the parabola.
13. Take a strip of paper or a ruler, and put the edge of the paper on the leftmost x -intercept so that the paper covers the portions of the graphs to the right of that intercept. For each of the linear functions still showing, are the y -coordinates positive or negative, that is, are the lines above (+) or below (–) the x -axis? _____
14. In the part of the parabola that is still showing, are the y -coordinates positive or negative? _____
15. Move your strip of paper to the right until you come to the next x -intercept. Examine the portions of the graph between the two x -intercepts. Are the y -coordinates for each of the linear functions in this section positive or negative? _____
16. Are the y -coordinates of the part of the parabola in this section of the Cartesian plane positive or negative? _____
17. Remove your strip of paper, and examine the portions of the linear functions to the right of the second x -intercept. Are the y -coordinates for each of the linear functions in this section positive or negative? _____
18. Are the y -coordinates of the part of the parabola in this section of the Cartesian plane positive or negative? _____
19. What can you conclude about the relationship between the signs of the y -coordinates of the linear functions and the signs of the y -coordinates on the parabola? _____
20. Without actually multiplying the linear expressions to obtain a quadratic expression, sketch the quadratic function that would be obtained by taking the products of the linear expressions given by the lines.

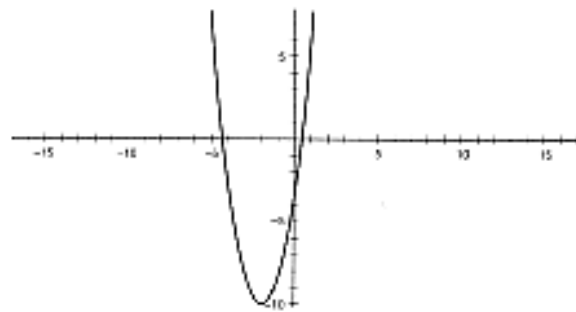


21. Look at questions 10 to 19 for this new combination of graphs. Do the relationships that you described when you answered these questions hold for the graph in question 20? Explain.
22. What characteristics of the two lines will cause the parabola to be inverted?

WORKING BACKWARD

SHEET 2

1. Given the parabola at the right, sketch lines that could represent its linear factors.



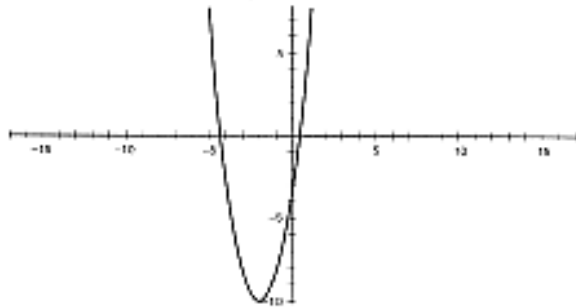
2. Did you use both the x -intercepts and the y -intercept of the parabola to make your choices? Explain.

3. What else did you consider while making your choices?

4. Write the linear equations for these lines. _____

5. Given these linear equations, what would be the equation of the parabola? _____

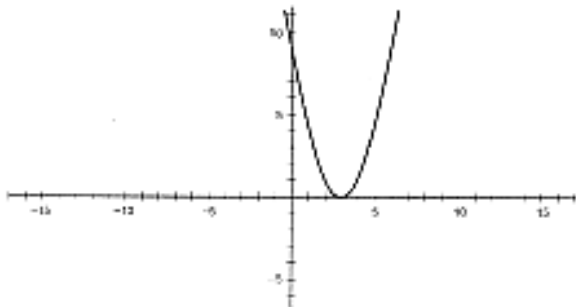
6. Would another pair of lines work? If so, show another set of lines on the graph at right.



7. Given these linear equations, what would be the equation of the parabola? _____

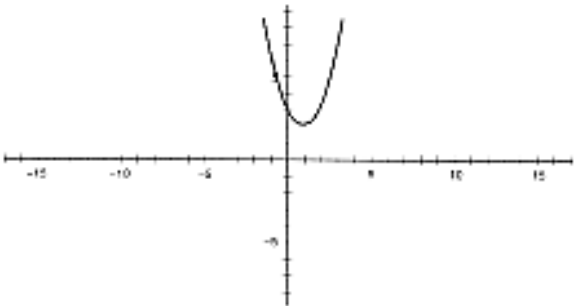
8. What do you notice regarding the product of the linear factors?

9. The quadratic function at right has only one x -intercept. Sketch lines representing its linear components.



10. Explain your reasoning.

11. Try to sketch lines that are components of the quadratic function at right.



12. What did you find?

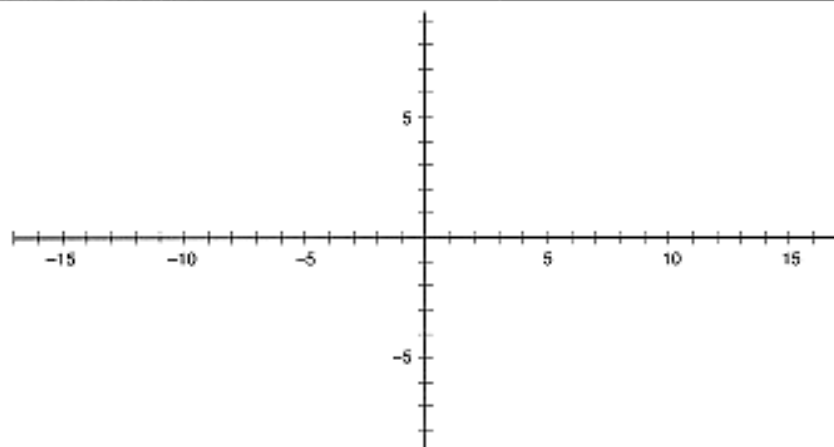
13. What is the reason for what you found?

POLYNOMIALS OF DEGREE THREE

SHEET 3

1. Write the equations for three (nonconstant) linear functions on the following lines, and graph these lines on the axes provided. Make sure that the equations that you choose are such that their graphs will be visible on the axes. Use three different colors to sketch the lines.

- a) _____
 b) _____
 c) _____

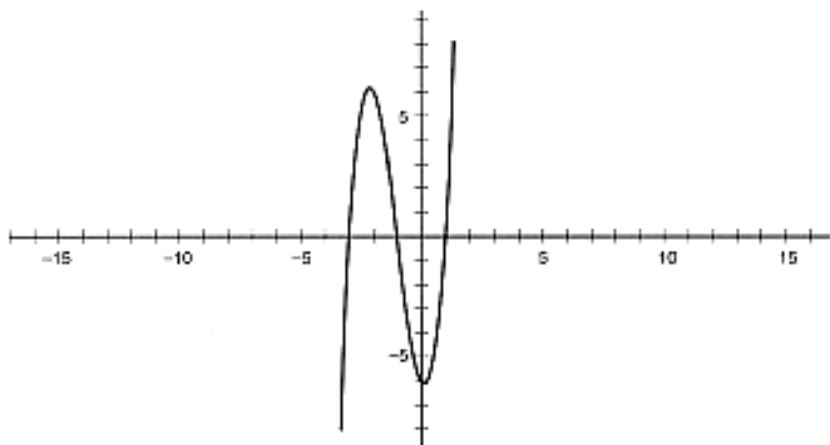


2. If you multiply the three linear expressions together, what type of expression do you obtain?

3. To predict the appearance of the graph of the new function built by multiplying these three expressions, sketch it on the axes in question 1.

4. Explain your reasoning.

5. Work backward—try to find three lines that could be components for the cubic polynomial at right.



6. Explain your reasoning.

ANSWERS to Building Polynomial Function

Sheet 1

1. $y = (-2/3)x + 2$ or an equivalent form.
2. Answers will vary. For example, students can use the slope-intercept form or the point-slope form to find the equation.
3. $y = (-2/3)x + 2$.
4. $y = (-2/3)(x - 3)$.
5. Lines will vary.
6. $-b/m$ is the x-intercept.
7. Answers will vary.
- 8.-10. Answers will vary. Students may not immediately see that the product of the two linear expressions gives a parabolic expression, nor do they necessarily recognize that the parabola and the lines share the same x- intercepts. Questions 11 and 12 are meant to highlight the intercepts for those students who have not made the connections.
11. The lines have the same x-intercepts as the parabola.
12. The y-intercept of the parabola is the product of the y-intercepts of the lines.
- 13.-18. Answers will vary. The teacher should assist students who are placing their paper strip incorrectly.
19. For any section of the graph, the product of the signs of the y-coordinates of the linear functions is the same as the sign of the y-coordinates of the parabola in that section.
20. A sample graph is given below. The parabola should have the same x- intercepts as the lines have, and the y-intercept should be 3. It is inverted.
21. Yes. Explanations will vary, but students should state again at least some of the relationships among the graphs found in questions 10 to 20.
22. One line will have a positive slope, and the other will have a negative slope.

Sheet 2

1. The lines drawn should go through the x-- intercepts of the parabola. The product of the y-intercepts of the lines should equal the y-- intercept of the parabola. The parabola should have negative y-coordinates when just one of the lines has negative y-coordinates. A possible pair of lines is given below.
2. Students may not have considered both the x-- intercepts and the y- intercepts when sketching their graphs. They certainly could have used a combination of y-coordinates for a value of $x = 0$, but using the y-intercepts is easier.
3. Students should also consider the signs of the y-coordinates for various sections of the graph.
4. Equations will vary One possibility is $y = x + 4$ and $y = 2x - 1$. Another possibility is $y = (1/2)x$ and $y = 4x - 2$. For all equations, the x-intercepts should be -4 and $1/2$, and the product of the y—intercepts should be -4 .
5. The quadratic expression should be the product of the linear expressions given in question 4. If the linear expressions were $x + 4$ and $2x - 1$, then the quadratic function would be $y = 2x^2 + 7x - 4$.
6. Yes. See remarks in the text about generating other sets of lines by distributing unit factors.
7. The answer will be the same as the one for question 5.
8. Students should notice that the product is the same even though the lines have changed.

9.-10. The graph should show two lines, both having an x-intercept of 3. The product of the y--intercepts of the lines is 9. Two lines are needed, and since only one x-intercept exists, both lines must go through that intercept. The y-- intercepts of the lines must have a product of 9. Note that the two lines could both be $y = -x + 3$; could both be $y = x - 3$; or could differ, for example, $y = (1/3)x - 1$, and $y = 3x - 9$.

11.-12. No lines can be drawn that would be components of the quadratic function given.

13. The absence of x-intercepts implies that no real roots exist, that is, lines cannot be drawn, because the quadratic equation cannot be factored into linear expressions over the real numbers.

Sheet 3

1. Equations and graphs will vary.

2. A cubic expression

3.-4. The graph drawn should go through the x-- intercepts of all three lines. The product of the y--intercepts of the lines gives the y-intercept of the cubic. Help in graphing the cubic can also be obtained by observing the signs of the y-- coordinates.

5.-6. Each line should pass through one of the x-intercepts. The product of the y-intercepts of the lines should be -6. Exactly one or exactly three of the lines will have negative y-coordinates when the y-coordinates on the cubic are negative. One such set of lines would be $y = x - 1$, $y = -x - 3$, and $y = -2x - 2$.

Translations through Families

First enter the equation into the graphing screen by pushing **ON** then **Y=**. Your screen should look like:

```

Plot1 Plot2 Plot3
Y1=
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=

```

Into Y_1 enter an equation, the parent function for example $y=x$.

Move the cursor until the Y_1 is highlighted then push **X,T,θ,n** to put in the variable “X”. Then enter the function shifted up 2 by moving the cursor until Y_2 is highlighted and pushing **X,T,θ,n** **+|2**. Your screen should look like:

```

Plot1 Plot2 Plot3
Y1=X
Y2=X+2
Y3=
Y4=
Y5=
Y6=
Y7=

```

On the Y_1 line use the arrows to move the cursor all the way over to the left past Y_1 until the slash (/) is highlighted push **ENTER**. The slash line should be much thicker. This will make the parent function a heavy solid line and the translated function a lighter line so you can tell the difference. The screen should look like:

```

Plot1 Plot2 Plot3
Y1=X
Y2=X+2
Y3=
Y4=
Y5=
Y6=
Y7=

```

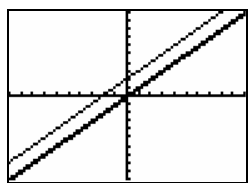
Push **ZOOM** to set the window and using the arrow keys move the cursor down until you reach option 6: which says “ZSTANDARD”

```

ZOOM MEMORY
1:ZBox
2:Zoom In
3:Zoom Out
4:ZDecimal
5:ZSquare
6:ZStandard
7:ZTrig

```

Push **ENTER** to select then push **GRAPH** to see the graph on a 10X10 screen. The graph should look like:

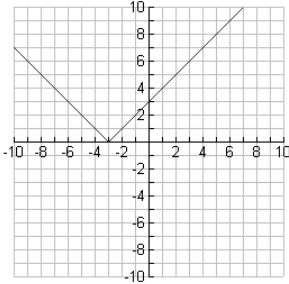


To graph another change with the parent function simply push **GRAPH** again, highlight Y2, and push **CLEAR**. Then enter the next function.

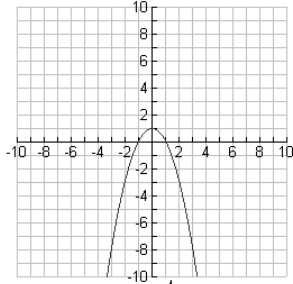
Mix and Match Functions

Name _____

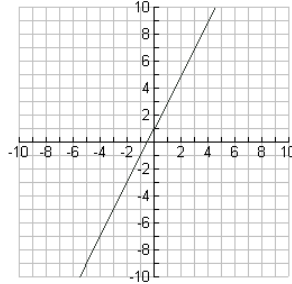
A



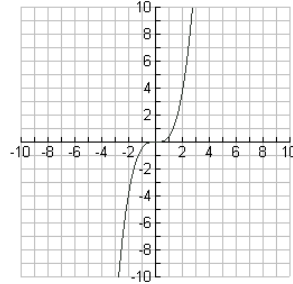
B



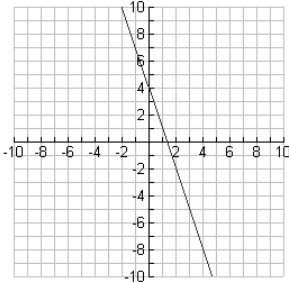
C



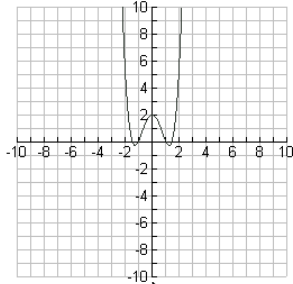
D



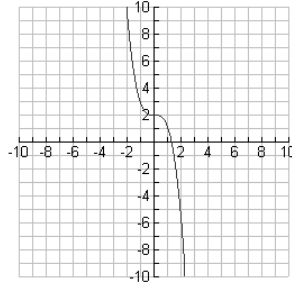
E



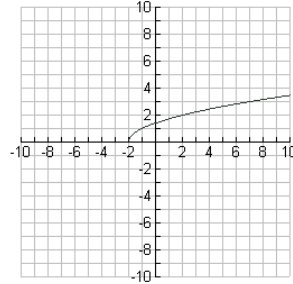
F



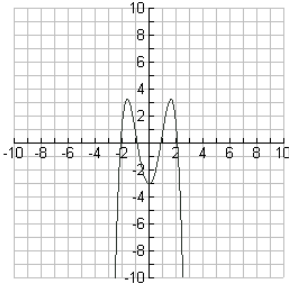
G



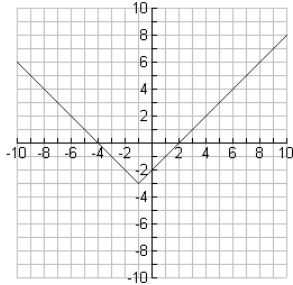
H



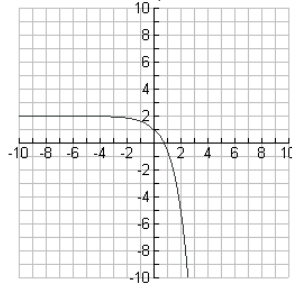
I



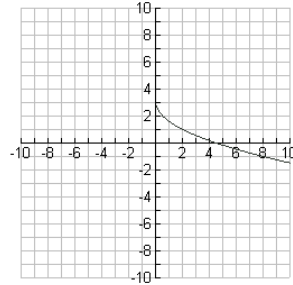
J



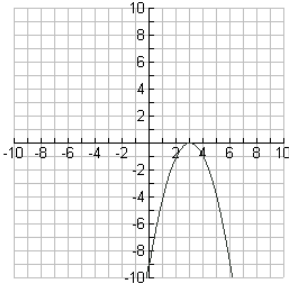
K



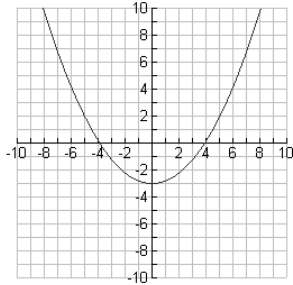
L



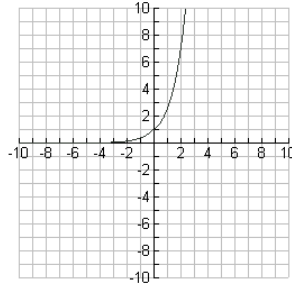
M



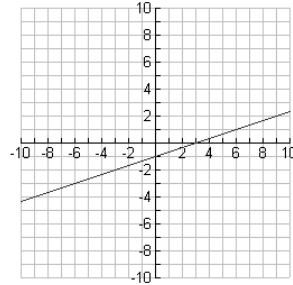
N



O



P



Which function belongs to each equation?

1. $y = \sqrt{x+2}$ _____

2. $y = |x+3|$ _____

3. $y = 2x+1$ _____

4. $y = -3x+4$ _____

5. $y = -x^2+1$ _____

6. $y = -(x-3)^2$ _____

7. $y = -\sqrt{2x}+3$ _____

8. $y = \frac{1}{2}x^3$ _____

9. $y = |x+1|-3$ _____

10. $y = e^x$ _____

11. $y = x^4 - 3x^2 + 2$ _____

12. $y = -x^3 + 2$ _____

13. $y = -x^4 + 5x^2 - 3$ _____

14. $y = \frac{1}{5}x^2 - 3$ _____

15. $y = -e^x + 2$ _____

16. $y = \frac{1}{3}x - 1$ _____

Which function belongs to each equation?

Key

1. $y = \sqrt{x+2}$ H

2. $y = |x+3|$ A

3. $y = 2x+1$ C

4. $y = -3x+4$ E

5. $y = -x^2+1$ B

6. $y = -(x-3)^2$ M

7. $y = -\sqrt{2x}+3$ L

8. $y = \frac{1}{2}x^3$ D

9. $y = |x+1|-3$ J

10. $y = e^x$ O

11. $y = x^4 - 3x^2 + 2$ F

12. $y = -x^3 + 2$ G

13. $y = -x^4 + 5x^2 - 3$ I

14. $y = \frac{1}{5}x^2 - 3$ N

15. $y = -e^x + 2$ K

16. $y = \frac{1}{3}x - 1$ P